A LAMBERT W FUNCTION APPROACH FOR SOLUTION OF SECOND ORDER DELAY DIFFERENTIAL EQUATION AS A SPECIAL CASE OF THE ONE-MASS SYSTEM CONTROLLED OVER THE NETWORK

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ABSTRACT

This paper is concerned with analyzing the transcendental characteristic equation (TCE) of a linear time invariant system with a single delay which has infinite number of roots that makes the stability of a time delay system extremely difficult. The basic concepts and results to the solutions as well as the analytical stability analysis of a second order delay differential equation which can be considered as a special case of the one mass controlled over the network was studied by an approach using the Lambert W function. Further the theoretical results are supported through the numerical examples.

Keywords: Lambert W function, Delay Differential Equation, Stability


1. INTRODUCTION

Time delay systems are often described by delay differential equations which belong to the class of functional differential equations. Time delay systems are systems in which a significant time delay exist between the applications of input to the system and their resulting effect. Recent research in different field such as Laser Physics, Electronic Engineering, Control Theory, Population Dynamics, etc., have shown that an important role of DDE in explaining many physical phenomena. Indeed, many physical events do not occur instantaneously, but can be modelled with delays.

In general a transcendental characteristic equation (TCE) in a linearized time invariant system with a single delay is of the form $A(s) + B(s)e^{-\tau s} = 0$, where $\tau$ is the delay time.
The analytical stability of a time delay system is affected by the infinite number of roots of TCE. Applied mathematicians and control engineers have a long impact effort on the analysis of DDE stability. Numerical methods, asymptotic solutions and graphical tools are the few methods for solving DDEs mainly for stability analysis. Under various conditions by solving characteristic equation an analytical solution of DDEs is found by several attempts.

Using Lambert function the stability analysis and the complete solution of a linear chatter problem is presented by Asl et al., in 2003 [3]. Using Lambert W function Yi et al., in 2012 [6] developed a technique for finding angular speed and position regulation. In 2011, Yi et al., derived stability conditions from the locations of the characteristic roots in the complex plane based on the Lambert W function approach [7]. Based upon the matrix Lambert W function method Yi et al., in 2007[8] developed the Laplace transform technique for solving linear systems of DDEs. Stability analysis of time-delay systems is studied as the application of Lambert W function by Hwang et al., in 2005 [1]. Using Lambert W function method the solution of the matrix differential equation with delayed argument is found by Ivanoviene et al., in 2013 [4]. In 2002, the analytical stability bound is found for delayed second-order systems with repeated poles by Chen et al.,[12].The method of finding the solutions of the scalar and matrix differential equations with the delayed argument is analysed based on application of the Lambert function by Ivanoviene et al., in 2008[5]. The summary on the recent research results on delay differential equations and its applications using the matrix Lambert W function was given by Yi et al., in 2008 [9, 11]. He et al., discussed the stability and state feedback stabilization of first-order linear time-delay system in detail via the Lambert function in 2011[10]. Bandopadhya proposed a solution technique to investigate and analyse the time-delay response of active material actuator applying Lambert W function in 2013[2].

This paper is organized as follows. In section 2, first the problem is considered as the dynamical control system whose governing equation can be written as second order delay differential equation with single delay. Detailed derivation for the proposed approach for obtaining solution of DDE as well as studying the analytical stability was explained, using Lambert W function. A few examples are illustrated in section 3.

2 SECOND ORDER DELAY DIFFERENTIAL EQUATION AND ITS STABILITY ANALYSIS

Consider the following second order Delay Differential Equation with constant co-efficient

$$\frac{d^2y(t)}{dt^2}+2\beta \frac{dy(t)}{dt} + \beta^2 y(t) = K_p y(t - \tau), t > 0$$

$$y(t) = \phi(t), t \in [-\tau, 0]$$

(2.1)

Where $\beta$, $K_p$ and $\tau$ are constants.

Here we are interested in showing the solution as well as stability analysis for the above equation which can be considered as the special case of the one-mass system controlled over the network whose governing equation is given by

$$\frac{d^2y(t)}{dt^2} + a_n \frac{dy(t)}{dt} + b_n y(t) = u(t)$$

(2.2)

Where $u(t)$ is a control input and $y(t)$ is a plant state.

In this system modelling the following delays are consider:

- $\tau^c$: The communication delay for certain protocol.
- $\tau^p$: The plant delay due to physical transportation delay intrinsic to the system under control.
The position \( y(t) \) has a time delay \( \tau \) which is the sum of \( \tau^c \) and \( \tau^p \) due to the communication network applied in the control system such as the web based manipulation of the mass position. Here the time delay \( \tau \) becomes a design parameter to shape the stability bound.

By using the proportional control law with constant proportional controller gain \( K_p \), i.e. \( u(t) = K_p y(t - \tau) \) (2.3)

Equation (2.2) becomes

\[
\frac{d^2 y(t)}{dt^2} + a_n \frac{dy(t)}{dt} + b_n y(t) - K_p y(t - \tau) = 0, \ t > 0
\]

\[
y(t) = \phi(t), \ t \epsilon [-\tau, 0]
\]

Where \( \phi(t) \) is the state’s initial function

Assume that

\[
b_n = \frac{(an)^2}{4} \triangleq \beta^2
\]

Let the solution of the Equation (2.4) is of the form \( y(t) = e^{st} \), where \( s \) is the complex number. Substituting it into Equation (2.4), we get

\[
(s + \beta)^2 - K_p e^{-st} = 0
\]

Which is the transcendental characteristic equation of the above DDE (2.4)

Multiplying \( e^{st} \) to both sides of the Equation (2.6) gives

\[
(s + \beta)^2 e^{st} = K_p
\]

Taking square root on both sides of the Equation (2.7)

\[
(s + \beta) e^{\left(\frac{t}{2}\right)s} = \pm \sqrt{K_p}
\]

Consider

\[
s_1 = \left(\frac{1}{2}\right)s; \ \beta_1 = \left(\frac{1}{2}\right)\beta; \ K_1 = \left(\frac{1}{2}\right)\sqrt{K_p}
\]

And the Equation (2.8) becomes

\[
(s_1 + \beta_1)e^{(s_1+\beta_1)t} = \pm K_1 e^{\beta_1}; \quad (2.9)
\]

Then using Lambert function (every function \( W(Z) \) satisfies \( W(Z)e^{W(Z)} = Z \) is a Lambert W function.) the transcendental characteristic equation (2.6) is solved as

\[
W(\pm K_1 e^{\beta_1}) = s_1 + \beta_1
\]

\[
s_1 = W(\pm K_1 e^{\beta_1}) - \beta_1
\]

\[
\frac{\tau}{2} s = W\left(\pm \frac{\tau}{2}\sqrt{K_p} e^{\left(\frac{t}{2}\right)\beta}\right) - \frac{\tau}{2} \beta
\]

\[
s = \frac{2}{\tau} W\left(\frac{\tau}{2} \left(\pm \sqrt{K_p}\right) e^{\left(\frac{t}{2}\right)\beta}\right) - \beta
\]

Here the characteristic root \( s \) is obtained analytically in terms of the parameters \( \tau, \beta \) (in terms of \( a_n, b_n \)) which affect the stability property of DDE (2.4).

In particular for a given system, parameters \( a_n \) and \( b_n \), we intend to study the solution as well as stability of the system corresponding to the preshape function \( \phi(t) \).

\[
y(t) = \sum_{k= -\infty}^{\infty} C_k e^{skt}
\]

Where, \( s_k = \frac{2}{\tau} W_k\left(\frac{\tau}{2} e^{\left(\frac{t}{2}\right)\beta} \left(\pm \sqrt{K_p}\right)\right) - \beta
\]

(2.11)
A Lambert W Function Approach for Solution of Second Order Delay Differential Equation as a Special Case of the One-Mass System Controlled Over the Network

\[ y(t) = \sum_{k=-\infty}^{\infty} C_k e^{\left(\frac{2W_k}{\beta} \left(\frac{t}{\beta} \right) \left(\pm \sqrt{k + \alpha}\right\}t\right)} \]  

(2.12)

Which is the general solution of the Equation (2.1). Where the co-efficient \( C_k \) \((k = 0, \pm 1, \pm 2, \pm 3 \ldots)\) Can be obtained using the preshape function \( \phi(t) \)

Since \( y(t) = \phi(t), t \in [-\tau, 0] \) we get

\[ \phi(t) = \sum_{k=-\infty}^{\infty} C_k e^{\left(\frac{2W_k}{\beta} \left(\frac{t}{\beta} \right) \left(\pm \sqrt{k + \alpha}\right\}t\right)} \quad t \in [-\tau, 0] \]

The procedure for finding the coefficient \( C_k \) as follows:

First divide the interval \([-\tau, 0]\) into \(2N\) parts. (here \(N\) is sufficiently large natural number).

\[ [-\tau, 0] = [-\tau, -\tau + \frac{\tau}{2N}] \cup [-\tau + \frac{\tau}{2N}, -\tau + \frac{2\tau}{2N}] \cup \cdots \cup [-\tau, 0] \]

Then find out the values of the preshape function \( \phi(t) \) at the endpoints of the separate intervals.

\[ \phi(t) = \phi(0) + \phi(\frac{\tau}{2N}) + \cdots + \phi(-\tau) \]

\[ \phi(t) = \phi(0) + \phi(\frac{\tau}{2N}) + \cdots + \phi(-\tau) \]

Now we get the system of \(2N+1\) equations.

\[ \begin{pmatrix} \phi(0) \\ \phi(-\frac{\tau}{2N}) \\ \vdots \\ \phi(-\tau) \end{pmatrix} = \begin{pmatrix} y_N(0) & \cdots & y_N(0) \\ \vdots & \ddots & \vdots \\ y_N(-\tau) & \cdots & y_N(-\tau) \end{pmatrix} \begin{pmatrix} C_N \\ C_{N+1} \\ \vdots \\ C_N \end{pmatrix} \]

\[ \phi(t, N) \approx Y(t, N)C(N) \]

\[ C(N) = Y^{-1}(t, N)\phi(t, N) \]

The coefficient \( C_k \) is given by following relation

\[ C_k = \lim_{N \to \infty} (Y^{-1}(t, N)\phi(t, N))_k \]

Here to find the preshape function \( \phi(t) \), we assume that \( y(t) = 0, t < -\tau \), then \( y(t-\tau) = 0, t \in [-\tau, 0] \).

Then the Equation (2.4) becomes the second order ODE.

\[ \frac{d^2y(t)}{dt^2} + a_n \frac{dy(t)}{dt} + b_n y(t) = 0 \]  

(2.13)

2.1 Stability Analysis

For system (2.1) to be stable, all the characteristic roots must have negative real part. In the general case, the characteristic roots \( s_k, k = 0, \pm 1, \pm 2, \ldots \) of the Equation (2.1) are obtained by solving the characteristic Equation (2.6), where \( s_k \) is a complex number. If the characteristic roots have negative real part, i.e. \( \text{Re}(s_k) < 0 \) for all \( k = 0, \pm 1, \pm 2, \ldots \) then the solution of (2.1)
is asymptotically stable and if at least one of the characteristic roots have positive real part, i.e. 
\(\text{Re}(\lambda_k) > 0\) for some \(k = 0, \pm 1, \pm 2, \ldots\), then the solution of (2.1) is unstable. Also the stability question is solved by the dominant characteristic roots of the system with the branches of the Lambert W function corresponding to \(k = 0, \pm 1, \pm 2, \ldots\). It is very difficult to calculate the solution for all the branches. In this case, if the values of the real part of the dominant eigenvalues is in the left half plane, then the system (2.1) is stable.

3. NUMERICAL EXAMPLE

3.1 Example 1

Consider the following second order DDE

\[
\frac{d^2y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + y(t) = 0.2y(t-1), \ t > 0
\]

(3.1)

\[y(t) = \phi(t), \ \text{te}[-1,0]\]

Here \(K_p=0.2\); \(\tau=1\) and \(\beta=1\).

To consider the stability margin, we restrict the maximal \(K_p\) by \(\tau_{\text{max}}\), since \(\beta=1\), by (2.5), \(b_n=1\) and \(a_n=2\).

Assume \(y(t) = 0, \ t < -1\) then, \(y(t-1) = 0, \ \text{te}[-1,0]\).

Therefore the Eqn. (3.1) becomes the second order ODE.

\[
\frac{d^2y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + y(t) = 0
\]

(3.2)

The preshape function of the above second order DDE (3.1) is obtained by solving the above ODE (3.2). Then the solution of the ODE is

\[y(t) = (A t + B)e^{-t}\]

(3.3)

Assuming \(y(0) = 0\) and \(y(0.1) = 1\), we get \(A=11.0497\) and \(B=0\). Hence the preshape function of the Eqn. (3.1) is

\[\phi(t) = (11.0497t)e^{-t}, \ \text{te}[-1,0]\]

Then the solution of second order DDEs (3.1) is

\[y(t) = \sum_{k=-\infty}^{\infty} C_k e^{(2W_k\left(\frac{1}{2}e^{2\pi(\sqrt{1/2})}\right)-1)t}\]

Where the unknown co-efficient \(C_k\) can be obtained as follows.

3.1.1 Case (i)

\[y(t) = \sum_{k=-\infty}^{\infty} C_k e^{(2W_k\left(\frac{1}{2}e^{2\pi(-1/2)}\right)-1)t}\]

To find the co-efficients \(C_k\), divide the interval [-1,0] into 2 parts.

\([-1,0] = [-1, -\frac{1}{2}] U [-\frac{1}{2}, 0]\)

Here

\[\phi(0) = C_{-1}y_{-1}(0) + C_0y_0(0) + C_1y_1(0)\]

\[\phi\left(-\frac{1}{2}\right) = C_{-1}y_{-1}(-\frac{1}{2}) + C_0y_0(-\frac{1}{2}) + C_1y_1(-\frac{1}{2})\]

\[\phi(-1) = C_{-1}y_{-1}(-1) + C_0y_0(-1) + C_1y_1(-1)\]

The approximation of this system in matrix form as follows
A Lambert W Function Approach for Solution of Second Order Delay Differential Equation as a Special Case of the One-Mass System Controlled Over the Network

\[
\begin{align*}
&\begin{pmatrix}
\phi(0) \\
\phi(-\frac{1}{2}) \\
\phi(-1)
\end{pmatrix} = 
\begin{pmatrix}
y_{-1}(0) & y_{0}(0) & y_{1}(0) \\
y_{-1}(-\frac{1}{2}) & y_{0}(-\frac{1}{2}) & y_{1}(-\frac{1}{2}) \\
y_{-1}(-1) & y_{0}(-1) & y_{1}(-1)
\end{pmatrix}
\begin{pmatrix}
C_{-1} \\
C_{0} \\
C_{1}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&\begin{pmatrix}
C_{-1} \\
C_{0} \\
C_{1}
\end{pmatrix} = 
\begin{pmatrix}
y_{-1}(0) & y_{0}(0) & y_{1}(0) \\
y_{-1}(-\frac{1}{2}) & y_{0}(-\frac{1}{2}) & y_{1}(-\frac{1}{2}) \\
y_{-1}(-1) & y_{0}(-1) & y_{1}(-1)
\end{pmatrix}^{-1}
\begin{pmatrix}
\phi(0) \\
\phi(-\frac{1}{2}) \\
\phi(-1)
\end{pmatrix}
\end{align*}
\]

\[
\begin{pmatrix}
1 \\
-11.5650 - 18.5851i \\
-2.1166e + 02 + 4.2987e + 02i
\end{pmatrix}
\begin{pmatrix}
1 \\
12.474 \\
1.556
\end{pmatrix}
\begin{pmatrix}
1 \\
11.5650 + 18.5851i \\
-2.116e + 02 - 4.2987e + 02i
\end{pmatrix}
\begin{pmatrix}
0 \\
-3.3510 \\
4.0652
\end{pmatrix}
\]

Therefore the solution of the Eqn.(3.1) is given as

\[
y(t) = (0.0721 - 0.0405i)e^{-6.1720 + 8.3115t} + (-0.1441 + 0.0000i)e^{-0.4421t}
\]

\[
+ (0.0721 + 0.0405i)e^{-6.1720 + 8.3115t}
\]

The characteristic roots s for the branch \( k = 0, \pm 1, \pm 2, \pm 3 \) as follows:

\[
\begin{align*}
k = 0; \quad & s = -0.4421 \\
k = 1; \quad & s = -6.1720 + 8.3115i \\
k = -1; \quad & s = -6.1720 - 8.3115i \\
k = 2; \quad & s = -7.8309 + 21.3724i \\
k = -2; \quad & s = -7.8309 - 21.3724i \\
k = 3; \quad & s = -8.7187 + 34.1125i \\
k = -3; \quad & s = -8.7187 - 34.1125i
\end{align*}
\]

Here principal branch (k = 0) has only real root. For \( k = \pm 1, \pm 2, \pm 3 \), characteristic roots are complex conjugate. Also all the characteristic roots have negative real part. Therefore the system is stable.

### 3.1.2 Case (ii)

\[
y(t) = \sum_{k=-\infty}^{\infty} C_k e^{2W_k \left(\frac{1}{2} e^{\sqrt{0.2}}\right)-1} t
\]

\[
\begin{pmatrix}
C_{-1} \\
C_{0} \\
C_{1}
\end{pmatrix} = 
\begin{pmatrix}
y_{-1}(0) & y_{0}(0) & y_{1}(0) \\
y_{-1}(-\frac{1}{2}) & y_{0}(-\frac{1}{2}) & y_{1}(-\frac{1}{2}) \\
y_{-1}(-1) & y_{0}(-1) & y_{1}(-1)
\end{pmatrix}^{-1}
\begin{pmatrix}
\phi(0) \\
\phi(-\frac{1}{2}) \\
\phi(-1)
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
4.4658 + 0.2921i \\
19.8578 + 2.6091i
\end{pmatrix}
\begin{pmatrix}
1 \\
4.4658 - 0.2921i \\
19.8578 - 2.6091i
\end{pmatrix}
\begin{pmatrix}
1 \\
13.8039 - 33.3700i \\
-9.2301e + 02 - 9.2127 + 0 2v
\end{pmatrix}
\begin{pmatrix}
0 \\
-3.3510 \\
4.0652
\end{pmatrix}
\]

Therefore the solution is
The characteristic roots \( s \) for the branch \( K = \pm 1, \pm 2, \pm 3 \) as follows:

\[
\begin{align*}
K = 0; & \quad s = -2.9972 + 0.1306i \\
K = 1; & \quad s = -7.1733 + 14.9235i \\
K = -1; & \quad s = -2.9972 - 0.1306i \\
K = 2; & \quad s = -8.3238 + 27.7584i \\
K = -2; & \quad s = -7.1733 - 14.9235i \\
K = 3; & \quad s = -9.0483 + 40.4479i \\
K = -3; & \quad s = -8.3238 + 27.7584i
\end{align*}
\]

In this case there is no real root exist at the principal branch \( K = 0 \). But the complex conjugate roots exist for \( K = (0,-1), (1,-2), (2,-3) \) and so on. Also all the characteristic roots have negative real part. Therefore the system is stable.

### 3.2 Example 2

Consider the following second order DDE

\[
\frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 4y(t) = 1.5y(t - 2), \quad t > 0
\]

(3.4)

\[y(t) = \phi(t), \quad t \in [-2,0]\]

Here \( K_p=1.5; \tau=2 \) and \( \beta=2 \). Since \( \beta=2 \), \( b_n=4 \) and \( a_n=4 \).

Assume \( y(t) = 0 \); \( t < -2 \), then \( y(t - 2) = 0 \), \( t \in [-2,0] \). Therefore the Equation (3.4) becomes the second order ODE

\[
\frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 4y(t) = 0
\]

(3.5)

The preshape function of the above second order DDE (3.4) is obtained by solving the above ODE (3.5). Then the solution of the ODE is

\[
y(t) = (At + B)e^{-2t} \tag{3.6}
\]

Assuming \( y(0) = 0 \) and \( y(1) = 1 \). We get A=7.3910 and B=0. Hence the preshape function of the Eqn.(3.4) is

\[
\phi(t) = (7.3910 t)e^{-2t}, \quad t \in [-2,0].
\]

Then the solution of second order DDEs (3.4) is

\[
y(t) = \sum_{k=-\infty}^{\infty} C_k e^{\left(W_k(e^{2(\sqrt{15})})-2\right)t}
\]

#### 3.2.1 Case (i)

\[
y(t) = \sum_{k=-\infty}^{\infty} C_k e^{\left(W_k(e^{2(\sqrt{15})})-2\right)t}
\]

To find the co-efficient \( C_k \), divide the interval \([-2,0]\) into two parts.

\([-2, 0] = [-2,-1] \cup [-1,0]\)

Now,

\[
\phi(0) = C_{-1}y_{-1}(0) + C_0y_0(0) + C_1y_1(0)
\]

\[
\phi(-1) = C_{-1}y_{-1}(-1) + C_0y_0(-1) + C_1y_1(-1)
\]

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A Lambert W Function Approach for Solution of Second Order Delay Differential Equation as a Special Case of the One-Mass System Controlled Over the Network

\[ \phi(-2) = C_{-1}y_{-1}(-2) + C_0y_0(-2) + C_1y_1(-2) \]

\[ \begin{pmatrix} \phi(0) \\ \phi(-1) \\ \phi(-2) \end{pmatrix} = \begin{pmatrix} y_{-1}(0) & y_0(0) & y_1(0) \\ y_{-1}(-1) & y_0(-1) & y_1(-1) \\ y_{-1}(-2) & y_0(-2) & y_1(-2) \end{pmatrix} \begin{pmatrix} C_{-1} \\ C_0 \\ C_1 \end{pmatrix} \]

\[ \Rightarrow \begin{pmatrix} C_{-1} \\ C_0 \\ C_1 \end{pmatrix} = \begin{pmatrix} y_{-1}(0) & y_0(0) & y_1(0) \\ y_{-1}(-1) & y_0(-1) & y_1(-1) \\ y_{-1}(-2) & y_0(-2) & y_1(-2) \end{pmatrix}^{-1} \begin{pmatrix} \phi(0) \\ \phi(-1) \\ \phi(-2) \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 1 & 1 \\ 0.5045 - 3.9513i & 1.3737 & 0.5045 + 3.9513i \\ -15.3585 - 3.9868i & 1.8871 & -15.3585 + 3.9868i \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -54.6125 \\ -807.0699 \end{pmatrix} \]

\[ = \begin{pmatrix} 22.9698 - 1.8578i \\ -45.9396 + 0.0000i \\ 22.9698 + 1.8578i \end{pmatrix} \]

Therefore the solution of the Equation (3.4) is given as

\[ y(t) = (22.9698 - 1.8578i)e^{-1.3821 - 4.8394t} + (-45.9396 + 0.0000i)e^{-0.3175t} + (22.9698 + 1.8578i)e^{-1.3821 + 4.8394t} \]

The characteristic roots s for the branch \( k = 0, \pm 1, \pm 2, \pm 3 \) as follows:

- \( k = 0; \quad s = -0.3175 \)
- \( k = 1; \quad s = -1.3821 + 4.8394i \)
- \( k = -1; \quad s = -1.3821 - 4.8394i \)
- \( k = 2; \quad s = -2.1933 + 10.9780i \)
- \( k = -2; \quad s = -2.1933 - 10.9780i \)
- \( k = 3; \quad s = -2.6453 + 17.2414i \)
- \( k = -3; \quad s = -2.6453 - 17.2414i \)

Here principal branch (\( k = 0 \)) has only real root. For \( k = \pm 1, \pm 2, \pm 3 \), characteristic roots are complex conjugate. Also all the characteristic roots have negative real part. Therefore the system is stable.

### 3.2.2 Case (ii)

\[ y(t) = \sum_{k=-\infty}^{\infty} c_k e^{(\text{W}_k(\text{e}^{2(\sqrt{13})}) - 2)t} \]

\[ \begin{pmatrix} C_{-1} \\ C_0 \\ C_1 \end{pmatrix} = \begin{pmatrix} y_{-1}(0) & y_0(0) & y_1(0) \\ y_{-1}(-1) & y_0(-1) & y_1(-1) \\ y_{-1}(-2) & y_0(-2) & y_1(-2) \end{pmatrix}^{-1} \begin{pmatrix} \phi(0) \\ \phi(-1) \\ \phi(-2) \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 1 & 1 \\ -1.0561 + 1.7300i & -1.0561 - 1.7300i & -0.1137 - 6.4272i \\ -1.8777 - 3.6543i & -1.8777 + 3.6543i & -41.2959 + 1.4621i \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -54.6125 \\ -807.0699 \end{pmatrix} \]

\[ = \begin{pmatrix} 26.6233 + 24.8163i \\ -46.7586 - 27.4298i \end{pmatrix} \]

Therefore the solution is

\[ y(t) = (26.6233 + 24.8163i)e^{(-0.7065 - 2.1189)t} + (-46.7586 - 27.4298i)e^{(-0.7065 + 2.1189)t} + (20.1353 + 2.6135i)e^{(-0.7065 + 2.1189)t} \]
The characteristic roots $s$ for the branch $k = 0, \pm 1, \pm 2, \pm 3$ as follows:

- $k = 0; \quad s = -0.7065 + 2.1189i$
- $k = 1; \quad s = -1.8607 + 7.8717i$
- $k = -1; \quad s = -0.7065 - 2.1189i$
- $k = 2; \quad s = -2.4443 + 14.1057i$
- $k = -2; \quad s = -1.8607 - 7.8717i$
- $k = 3; \quad s = -2.8126 + 20.3805i$
- $k = -3; \quad s = -2.4443 - 14.1057i$

In this case there is no real root exist at the principal branch $k = 0$. But the complex conjugate roots exist for $k = (0,-1), k = (1,-2), k = (2,-3)$ and so on. Also all the characteristic roots have negative real part. Therefore the system is stable.

4 CONCLUSIONS

In this paper we discussed the analytical solution of second order delay differential equation using Lambert W function. Also the analytic stability was explained and supported through examples. The work can be extended for second order delay differential equation with matrix co-efficient.

REFERENCES


A Lambert W Function Approach for Solution of Second Order Delay Differential Equation as a Special Case of the One-Mass System Controlled Over the Network


