MAJORIZATION PROBLEMS FOR SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING Q-CALCULUS OPERATOR

K.Thilagavathi, K.Vijaya and K.Uma
Department of Mathematics, School of Advanced Sciences,
VIT University, Vellore - 632014, India

ABSTRACT
In the present paper, we investigate majorization problems for certain classes of analytic functions involving Ruscheweyh q-differential operator.

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1. INTRODUCTION
Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic and univalent in the the open unit disk

\( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \)

We now give some basic definitions and related details of \( q \)-calculus which are relevant for our study.

The \( q \)-shifted factorial is defined for \( \alpha, q \in \mathbb{C} \) as a product of \( n \) factors by

\[
(\alpha; q)_n = \begin{cases} 1 & ; n = 0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & ; n \in \mathbb{N}, \end{cases}
\]

and in terms of the basic analogue of the gamma function

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\( (q^n; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n > 0), \)  

(3)

where the \( q \)-gamma function is defined by ([3], p. 16, eqn. 1.10.1))

\[ \Gamma_q(x) = \frac{(q; q)_x(1-q)^{1-x}}{(q^x; q)_x} \quad (0 < q < 1). \]

(4)

If \(|q| < 1\), the equation (2) remains meaningful for \( n = \infty \) as a convergent infinite product:

\[ (\alpha_q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j). \]

In view of the relation

\[ \lim_{q \to 1^-} \frac{(q^n; q)_n}{(1-q)^n} = (\alpha)_n, \]

(5)

we observe that the \( q \)-shifted factorial (2) reduces to the familiar Pochhammer symbol \((\alpha)_n\), where 

\[ (\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1). \]

The \( q \)-derivative and \( q \)-integral of a function on a subset of \( C \) are respectively given by (see [3], pp. 19–22)

\[ \partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} \quad (z \neq 0, q \neq 0) \]

(6)

and

\[ \int_{0}^{z} f(t) d(t; q) = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \]

(7)

Therefore, the \( q \)-derivative of \( f(z) = z^n \), where \( n \) is a positive integer, is given by

\[ \partial_q z^n = \frac{z^n - (nz) - (1-q)^n}{z(1-q)} = z^{n-1} \quad (z \neq 0, q \neq 0). \]

(8)

For any non-negative integer \( n \), the \( q \)-integer number \( n, [n] \) is defined by:

\[ [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}, [0] = 0. \]

(9)

The \( q \)-number shifted factorial is defined by \([0]! = 1\) and \([n]! = [1][2] \ldots [n] \). Here \( q \) to be a fixed number between 0 and 1.

As \( q \to 1 \) we have

\[ [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1} = 1. \]
Here we used the following relations

\[ [m+n]_q = [m]_q + q^n [n]_q \]

\[ [m-n]_q = q^{-n} [m]_q - q^n [n]_q, [0]_q = 0, [1]_q = 1 \]

Recently, many authors have introduced new classes of analytic functions using \( q \)-calculus operators and related topics, we refer the reader to [1, 6, 11] and the references cited therein.

Using the definition of Ruscheweyh differential operator [12] for \( f \in A \), Kanas and Raducanu[4] defined and discussed the Ruscheweyh \( q \)-differential as

\[ R_q^\lambda f(z) = f(z) * F_{\lambda,q+1}(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} a_n z^n. \] (10)

Making use of (10) and properties of Hadamard product, Kanas and Raducanu [4] obtain the following equality

\[ z \partial_q (R_q^\lambda f(z)) = (1+q^{-\lambda}) R_q^{\lambda+1} f(z) - \frac{[\lambda]}{q^\lambda} R_q^\lambda f(z). \] (11)

It is easily verified from (10) if \( q \to 1^- \) the equality (11) implies

\[ z(R_q^\lambda f(z)) = (1+\lambda) R_q^{\lambda+1} f(z) - \lambda R_q^\lambda f(z). \] (12)

Recently Selvakumaran et.al.[13] proved the following result analogues to Nehari [10] result.

**Lemma 1** [13] If \( f(z) \) is analytic and bounded in \( U \), then

\[ |D_q f(z)| = \frac{1-|f(z)|^2}{1-z f(z)}, (z \in U). \]

For two functions \( f(z) \) and \( g(z) \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \) written \( f(z) \prec g(z) \) if there exists a schwarz function \( \omega(z) \) which is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(\omega(z)) \), \( z \in U \).

Let \( f \) and \( g \) be two analytic functions in \( U \). We say that \( f(z) \) is majorized by \( g(z) \) in \( U \) and write \( f(z) \ll g(z) \) if there exists a function \( \phi(z) \), analytic in \( U \), such that \( |\phi(z)| < 1 \) and \( f(z) = \phi(z) g(z) \), \( z \in U \).

It is noted that the notation of majorization is closely related to the concept of quasi-subordination between analytic functions.

Majorization problem for the class of analytic functions had been investigated by MacGregor [5] and Altintas et al. [2] further by Murugusundaramoorthy et al., [7, 8] for certain classes of analytic functions involving linear operators. Motivated by the results given in Kanas and Raducanu [4] and Nehari [10] in this paper, we define the following new class of starlike functions of complex order involving Ruscheweyh \( q \)-differential operator and obtain majorisation result for \( f \in S_q^\lambda(b) \).

For functions \( f \in A \) we let \( S_q^\lambda(b) \), the class of starlike functions of complex order if
\[ \Re \left( 1 + \frac{1}{b} \left[ z \partial_q (R_q f(z)) \right] \right) > 0 \]

\[(z \in \mathbb{U}, b \in \mathbb{C} \setminus \{0\}, 0 < q < 1)\]

It can be seen that, by specializing the parameters, the class \( S^q(b) \), reduces to many known subclasses of analytic functions. For instance, if \( q \to 1 \) then

(i) \( S^0_1(b) = S(b) \), the class of starlike functions of complex order \( b \) (see [9])

(ii) \( S^0_1(1 - \alpha) = S(\alpha), (0 \leq \alpha < 1) \) the class of starlike functions of complex order \( \alpha \).

2. MAIN RESULT

**Theorem 2** Let the function \( f(z) \) be in the class \( A \) and suppose that \( g(z) \in S^q_1(b) \). If \( R_q^q f(z) \) is majorized by \( R_q^q g(z) \). If \( R_q^q g(z) \) in \( \mathbb{U} \), then

\[ |R_q^{q+1} f(z)| \leq |R_q^{q+1} g(z)|, \, (|z| \leq r) \]

where

\[ r_i = r_i(\lambda : b) = \frac{K - \sqrt{K^2 - 4(q^4 + [\lambda])(1(2b-1)q^4 - [\lambda])}}{2((2b-1)q^4 - [\lambda])} \]

where

\[ K = 3(q^2 + [\lambda]) + 1(2b-1)q^4 - [\lambda] \]

\((b \in \mathbb{C} \setminus 0, 0 < q < 1, \lambda \geq 0)\)

**Proof.** Let

\[ h(z) = 1 + \frac{1}{b} \left[ z \partial_q (R_q^q g(z)) \right] \]

Since \( g(z) \in S^q_1(b) \), we have \( \Re h(z) > 0 (z \in \mathbb{U}) \) and

\[ h(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, (\omega \in A) \]

where

\[ \omega(z) = c_1 z + c_2 z^2 + \ldots \]

and \( A \) denotes the well known class of bounded analytic functions in \( \mathbb{U} \) and satisfies the conditions

\[ \omega(0) = 0, \text{ and } |\omega(z)| \leq |z|, (z \in \mathbb{U}) \]
It follows from (15) and (16) that
\[
\frac{z \partial_q (R_q^f g(z))}{R_q^f g(z)} = \frac{1+(2b-1)\omega(z)}{1-\omega(z)}
\]  
(18)

In view of the identity (15), we have the following inequality from (18) by making some simple calculations
\[
|R_q^f g(z)| \leq \frac{(1+|z|)(q^z + |\lambda|)}{q^z + |\lambda| + ((2b-1)q^z - |\lambda|)|z|} |R_q^{z+1} g(z)|
\]  
(19)

Since $R_q^f f(z)$ is majorized by $R_q^f g(z)$ in $U$, there exists an analytic function $\phi(z)$, such that
\[
R_q^f f(z) = \phi(z)(R_q^f g(z))
\]  
(20)

Applying $q$ differentiation with respect to $z$ and then multiplying by $z$ we get
\[
z \partial_q (R_q^f f(z)) = z \partial_q (\phi(z))(R_q^f g(z)) + z \phi(z) \partial_q (R_q^f g(z))
\]  
(21)

Noting that the $\phi(z)$ is bounded in $U$ and using lemma (1), we obtain
\[
|\partial_q (\phi(z))| \leq \frac{1-|\phi(z)|^2}{1-|z|^2}
\]  
(22)

and using (19), (22) in (21) we have
\[
|R_q^{z+1} f(z)| \leq \left( \phi(z) + \frac{1-|\phi(z)|^2}{1-|z|^2} \right) \times
\]
\[
\frac{(q^z + |\lambda|)|z|}{(q^z + |\lambda| + ((2b-1)q^z - |\lambda|)|z|} |R_q^{z+1} g(z)|
\]  
(23)

\[
= \frac{-q^z + |\lambda|}r \rho r^2 + (1-r)((q^z + |\lambda|) + ((2b-1)q^z - |\lambda|)|z|)\rho + (q^z + |\lambda|)r |R_q^{z+1} g(z)|
\]  
(24)

\[
|z| = r \quad \text{and} \quad |\phi(z)| \leq \rho \quad (0 \leq \rho \leq 1)
\]

\[
|R_q^{z+1} f(z)| \leq \frac{\psi(\rho)}{(1-r)((q^z + |\lambda|) + ((2b-1)q^z - |\lambda|)\rho + (q^z + |\lambda|)r) |R_q^{z+1} g(z)|
\]  
(25)

where
\[
\psi(\rho) = -(q^z + |\lambda|)\rho r^2 + (1-r)((q^z + |\lambda|) + ((2b-1)q^z - |\lambda|)\rho + (q^z + |\lambda|)r
\]

Taking its maximum value at $\rho = 1$ with $r = r_q(\lambda : b)$ given by (14). Furthermore, if $0 \leq \sigma \leq r = r_q(\lambda : b)$, the function $\phi(\rho)$ defined by
\[
\phi(\rho) = -(q^z + |\lambda|)\sigma r^2 + (1-\sigma)((q^z + |\lambda|) + ((2b-1)q^z - |\lambda|)\sigma)\rho + (q^z + |\lambda|)\sigma
\]
is an increasing function on $(0 \leq \rho \leq 1)$ so that
\[ \varphi(\rho) \leq \varphi(1) = -(q^2 + [\lambda])\sigma + (1-\sigma)((q^2 + [\lambda]) + ((2b-1)q^2 - [\lambda])\sigma) + (q^2 + [\lambda])\sigma \]  
(26)

\((0 \leq \rho \leq 1); 0 \leq \sigma \leq r_i = r_q(\lambda : b)\) then setting \( \rho = 1 \) in (24), we conclude that holds true for \(|z| \leq r_q(\lambda : b)\). This completes the proof of Theorem (2).

REFERENCES


