IDEAL THEORY IN A-TERNARY SEMIGROUPS

Seetha Mani P
Department of Mathematics, K L E F, Vaddeswaram, Guntur, Andhra Pradesh, India

Sarala Y
Faculty of Mathematics, National Institute of Technology, Andhra Pradesh, India

Jaya Lalitha G
Department of Mathematics, K L E F, Vaddeswaram, Guntur, Andhra Pradesh, India

Anjaneyulu A
Faculty of Mathematics, V.S.R & N.V.R College, Guntur, Andhra Pradesh, India

ABSTRACT

In this paper, we investigate and characterize the class of A– ternary semigroup. A characterization of the Thierrin radical of proper ideal in a A– ternary semigroup is given.

Key words: A-Ternary, Algebraic, Mathematics


http://www.iaeme.com/IJCIET/issues.asp?JType=IJCIET&VType=9&IType=3

1. INTRODUCTION

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results. The theory of ternary algebraic system was introduced by D.H. Lehmer [7]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. In this paper, we will make a intensive study of the notions of prime, completely prime and primary ideals in commutative ternary semigroups. The notion of an A- ternary semigroup will be defined and a characterization of an A- ternary semigroup will be presented with the aid of these notions, further algebraic properties of the radical of an ideal in a A-ternary semigroup.

2. PRELIMINARIES

Definition 2.1: An ideal $P$ in the ternary semigroup $T$ is said to be prime if $A$, $B$ and $C$ are ideals of $T$ such that $ABC \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. 
**Definition 2.2:** An ideal $P$ in a ternary semigroup $T$ is said to be a completely prime if $a, b, c \in T$ such that $abc \in P$ then $a \in P$ or $b \in P$ or $c \in P$.

Prime ideals in commutative ternary semigroups can be characterized in the following way:

**Theorem 2.3** (8): If $P$ is a proper ideal in a commutative ternary semigroup $T$, then the following statements are equivalent:
- $P$ is prime
- $P$ is completely prime.

If $A, B$ and $C$ are ideals of $T$ such that $P \not\subseteq A$, $P \not\subseteq B$ and $P \not\subseteq C$ then $ABC \not\subseteq P$.

**Definition 2.4:** Let $S$ be a non-empty, multiplicatively closed subset $(t_1, t_2, t_3 \in S$ implies $t_1t_2t_3 \in S$) of a ternary semigroup $T$. An ideal $P$ of $T$ is said to be maximal w.r.t $T$ provided
- $P \cap S = \emptyset$; and
- If $A$ is an ideal of $T$ such that $P \not\subseteq A$ then $A \cap S \neq \emptyset$.

**Theorem 2.5:** Let $S$ be a non-empty multiplicatively closed subset of a ternary semigroup $T$. If $A$ is an ideal in $T$ such that $A \cap S = \emptyset$, then there exists an ideal $P$ of $T$ such that $A \subseteq P$ and $P$ is maximal w.r.t $S$.

**Proof:** Let $F$ denote the collection of all ideals in $T$ which contain $A$ and are disjoint from $S$. Since $A \in F$, it follows that $F \neq \emptyset$. Define a relation $\preceq$ on $F$ by $B_1 \preceq B_2$ if $B_1 \subseteq B_2$. $F$ is a partially ordered set under the relation $\preceq$. If $\{B_i\}_{i \in I}$ is a non-empty chain in $F$, then $\bigcup_{i \in I} B_i \in F$ and $B_i \preceq \bigcup_{i \in I} B_i$, for every $i \in I$. Thus, every non-empty chain in $F$ has an upper bound in $F$.

By Zorn’s lemma implies that $F$ has a maximal element. Such a maximal element satisfies the conclusion of the theorem.

**Theorem 2.6:** Let $S$ be a nonempty, multiplicatively closed sub set of a commutative ternary semigroup $T$ and let $P$ be an ideal of $T$. If $P$ is maximal w.r.t $S$, then $P$ is prime.

**Proof:** Let $A, B$ and $C$ be ideals in $T$ such that $P \not\subseteq A$; $P \not\subseteq B$ and $P \not\subseteq C$. Since $P$ is maximal w.r.t $S$, it follows that $A \cap S \neq \emptyset$; $B \cap S \neq \emptyset$ and $C \cap S \neq \emptyset$. Let $a \in A \cap S$, $b \in B \cap S$ and $c \in C \cap S$. Since $S$ is closed under multiplication, it follows that $abc \in (ABC) \cap S$. Since $P \cap S = \emptyset$, it follows that $ABC \not\subseteq P$. Theorem 2.3 implies $P$ is prime.

**Definition 2.7:** An ideal $M$ in a ternary semigroup $T$ is said to be maximal provided:
- $M \not\subseteq T$ and
- If $A$ is an ideal in $T$ such that $M \not\subseteq A$, then $A = T$.

For ternary semigroups with an identity, Definition 2.4 and 2.7 can be connected by the following.

**Theorem 2.8:** Let $M$ be an ideal in the ternary semigroup $T$. If $T$ has an identity $\{e\}$ then the following statements are equivalent:
- $M$ is maximal.
- $M$ is maximal w.r.t $\{e\}$.

**Proof:** (i) $\Rightarrow$ (ii) Since $M$ is maximal, it follows that $M \cap \{e\} = \emptyset$. Theorem 2.5 implies there exists an ideal $P$ in $T$ such that $P$ is maximal w.r.t $\{e\}$ and $M \subset P$. Clearly $P \cap \{e\} = \emptyset$ implies $P \not\subseteq T$. Since $M$ is maximal, it follows that $M = P$. Thus, $M$ is maximal w.r.t $\{e\}$.
(ii) $\Rightarrow$ (i) Assume $M$ is not maximal. Then there exists an ideal $A$ in $T$ such that $M \nsubseteq A \nsubseteq T$. Clearly $A \nsubseteq T$ implies $A \cap \{e\} = \emptyset$. Thus $M$ is not maximal w.r.t $\{e\}$, a contradiction.

The following theorem is an immediate consequence of theorem 2.6 and theorem 2.8.

**Theorem 2.9:** Let $T$ be a commutative ternary semigroup with an identity. If $M$ is a maximal ideal in $T$, then $M$ is prime.

The following example shows that a maximal ideal in a commutative ternary semigroup without an identity may not be prime.

**Example 2.10:** Let $T$ denote the ternary semigroup of positive odd integers with the usual multiplication. If $M = \{x \in T / x > 2\}$, then $M$ is a maximal ideal in $T$. Since $2 \notin M$ and $2 \cdot 2 \cdot 2 = 8 \in M$, it follows that $M$ is not prime.

### 3. A-TERNARY SEMIGROUPS

**Definition 3.1:** A ternary semigroup $T$ is said to be an $A$-ternary semigroup provided:

- $T$ is commutative; and
- Every proper ideal in $T$ is contained in a prime ideal of $T$.

**Theorem 3.2:** A commutative ternary semigroup $T$ is an $A$-ternary semigroup if and only if the complement of every proper ideal contains a non-empty multiplicatively closed set.

**Proof:** If $T$ is an $A$-ternary semigroup, it is clear that the complement of every proper ideal contains a non-empty multiplicatively closed set. Let $B$ be a proper ideal in $T$ and let $S$ be a non-empty multiplicatively closed subset $T \setminus B$. By Theorem 2.7 implies there exists an ideal $P$ in $T$ such that $B \subset P$ and $P$ is maximal w.r.t $T$. Theorem 2.6 implies $P$ is prime. Hence, $M$ is maximal.

**Corollary 3.3:** If $T$ is a commutative ternary semigroup with an identity $e$, then $T$ is an $A$-ternary semigroup.

**Proof:** If $B$ is a proper ideal in $T$, then $\{e\} \subset T \setminus B$.

The following examples will show there exists $A$-ternary semigroup that do not have an identity.

**Example 3.4:** Let $T$ be a ternary semigroup of positive odd integers where $abc = \min \{a, b, c\}$. Then $T$ does not have an identity, and every proper ideal in $T$ is prime.

**Definition 3.5:** Let $P$ be a proper ideal in an $A$-ternary semigroup $T$. The radical of $P$ is denoted by $\sqrt{P}$ and is defined to be the intersection of all the prime ideals in $T$ that contains $P$.

The following theorem is an immediate consequence of Definition 3.5.

**Theorem 3.6:** If $P$ is a proper ideal in an $A$-ternary semigroup $T$, then $\sqrt{P}$ is an ideal in $T$ and $P \subset \sqrt{P}$.

**Definition 3.7:** A proper ideal $P$ in an $A$-ternary semigroup $R$ is said to be semiprime if $P = \sqrt{P}$.

**Theorem 3.8:** If $P$ is a prime ideal in an $A$-ternary semigroup $T$, then $P$ is semiprime.
Ideal Theory in A-Ternary Semigroups

**Proof:** Theorem 3.6 implies \( P \subseteq \sqrt{P} \). Let \( \{B_i\}_{i \in I} \) be the collection of all prime ideals in \( T \) that contain \( P \). Clearly, \( P \subseteq \{B_i\}_{i \in I} \) and \( \sqrt{P} = \bigcap_{i \in I} B_i \subseteq P \). Thus \( P = \sqrt{P} \).

**Theorem 3.9:** If \( P \) is a proper ideal in an A-ternary semigroup \( T \), then \( \sqrt{P} = \{y \in T \mid \exists \ n \in \text{odd natural number} \ N \ni y^n \in P\} \).

**Proof:** Let \( y \in T \) such that for some odd positive integers \( n \) it is valid that \( y^n \in P \). Let \( B \) be a prime ideal in \( T \) contains \( P \). Since \( B \) is prime and \( y^n \in B \), it follows that \( y \in B \). Since \( B \) was an arbitrary prime ideal in \( T \) containing \( P \) it follows that \( y \in \sqrt{P} \). Conversely, assume there exists \( y \in \sqrt{P} \) such that \( y^n \notin P \), for any \( n \in \text{odd natural number} \ N \). Choose one such \( y \) and let \( S = \{y^n / n \in \text{odd natural number} \ N\} \). Since, \( S \) is a non-empty, multiplicatively closed set in \( T \) such that \( S \cap P = \emptyset \); by Theorem 2.5 implies that there exists an ideal \( B \) in \( T \) such that \( P \subseteq B \) and \( B \) is maximal w.r.t \( T \). Theorem 2.6 implies that \( B \) is prime. Since, \( B \cap S = \emptyset \) and \( y \in S \), it follows that \( y \in B \). Thus \( y \notin \sqrt{P} \), a contradiction.

**Corollary 3.10:** If \( P \) is a proper ideal in an A- ternary semigroup \( T \), then \( \sqrt{P} \) is semiprime.

**Proof:** It is clear that the radical of \( P \) is a proper ideal in \( T \). Theorem 3.6 implies \( \sqrt{P} \subseteq \sqrt{\sqrt{P}} \).

Let \( y \in \sqrt{\sqrt{P}} \). Theorem 3.9 implies that there exist \( n \in \text{odd natural number} \ N \) such that \( y^n \in \sqrt{P} \). Moreover, theorem 3.9 now implies that there exists an odd positive integer \( m \) such that \( (y^n)^m \in P \). Thus \( y^{nm} \in P \) and Theorem 3.9 implies \( y \in \sqrt{P} \). Hence, \( \sqrt{\sqrt{P}} \subseteq \sqrt{P} \). Therefore \( \sqrt{P} \) is semiprime.

**Corollary 3.11:** Let \( P \) and \( Q \) be proper ideals in an A- ternary semigroup \( T \). Then

- If \( P \subseteq Q \), then \( \sqrt{P} \subseteq \sqrt{Q} \).
- If \( P \cap Q \neq \emptyset \), then \( \sqrt{P \cap Q} = \sqrt{P} \cap \sqrt{Q} \).

**Proof:**

i) If \( y \in \sqrt{P} \), then there exists \( n \in \text{odd natural number} \ N \) such that \( y^n \in P \subseteq Q \). Hence, \( y \in \sqrt{Q} \).

ii) Since \( P \cap Q \neq \emptyset \), it is clear that \( P \cap Q \) is an ideal in \( T \).

Let \( y \in \sqrt{A \cap B} \), then there exists \( n \in \text{odd natural number} \ N \) such that \( y^n \in A \cap B \). Therefore \( y^n \in A \) and \( y^n \in B \) and it follows that \( y \in \sqrt{A} \) and \( y \in \sqrt{B} \). Hence, \( y \in \sqrt{A \cap B} \). Consequently \( y \in \sqrt{A \cap B} \) implies that there exist \( n, m \in \text{odd natural number} \ N \) such that \( x^n \in A \) and \( x^m \in B \). Clearly \( x^{nm} \in A \cap B \). Thus \( x \in \sqrt{A \cap B} \).

Theorem 3.9 gives an important characterisation of the radical of an ideal in an A – ternary semigroup. Another characterization of the radical of an ideal will be developed. With the aid of this characterization, the concepts of the radical of an ideal in an A – ternary semigroup and of the Thierrin radical of an ideal can be connected.
Definition 3.12: Let $P$ be an ideal in a ternary semigroup $T$. A prime ideal $M$ in $T$ will be called a minimal prime divisor of $P$ provided.

- $P \subseteq M$
- If $C$ is prime ideal in $T$ such that $P \nsubseteq C \subseteq M$ then $C = M$.

Similarly completely prime ideals.

Theorem 3.13: If $P$ is a proper ideal in an $A$ – ternary semigroup $T$, then there exist a minimal prime divisor of $P$. Moreover, if $B$ is a prime ideal in $T$ such that $P \subset B$, then there exists a minimal prime divisor $M$ of $P$ such that $P \subseteq M \subset B$.

Proof: Since $P$ is a proper ideal in an $A$ – ternary semigroup $T$, it follows that there exists a prime ideal in $T$ containing $P$. Pick any such ideal and call it $B$. Let $F = \{ C \subset B | P \subset C \text{ and is a prime ideal in } T \}$. Define the relation $\leq$ on $F$ by $C_1 \leq C_2 \iff C_2 \subset C_1$. $F$ is a partially ordered set under the relation $\leq$. Let $\{ C_i \}_{i \in I}$ be a non-empty chain in $F$. Clearly, $\bigcap_{i \in I} C_i$ is an ideal in $T$. It is clear that $P \subset \bigcap_{i \in I} C_i \subset B$. It will follow that $\bigcap_{i \in I} C_i$ is an element in $F$ if it can be shown that $\bigcap_{i \in I} C_i$ is prime. Let $a, b, c \in T$ such that $abc \in \bigcap_{i \in I} C_i$. Clearly, $abc \in C_i$ for any $i \in I$. Suppose that there exist an $i_0 \in I$ such that $a \notin C_{i_0}, b \notin C_{i_0}$. Since $C_{i_0}$ is prime, it is clear that $c \in C_{i_0}$. Let $c_i \in \{ C_i \}_{i \in I}$. If $C_i \subset C_{i_0}$, then $c \in C_i$; otherwise, $C_i$ prime implies $a, b \in C_i \subset C_{i_0}$, a contradiction. If $C_i \subset C_i$, it is clear that $c \in C_i$. Since, $\{ C_i \}_{i \in I}$ is a chain in $F$, it follows that $c \in C_i$ for any $i \in I$. i.e., $c \in \bigcap_{i \in I} C_i$. Consequently, $\bigcap_{i \in I} C_i$ is prime. Therefore $\bigcap_{i \in I} C_i \in F$ and $\bigcap_{i \in I} C_i$ is an upper bound of the chain $\{ C_i \}_{i \in I}$. Zorn’s lemma implies that $F$ has a maximal element. Pick one such element and call it $M$. Let $D$ be a prime ideal in $T$ such that $P \nsubseteq D \subset M$. Clearly, $D \in F$ and $M \leq D$. Since $M$ is maximal in $F$, it follows that $D = M$. Therefore, $M$ is a minimal prime divisor of $B$.

Corollary 3.14: If $P$ is a proper ideal in an $A$ – ternary semigroup $T$, then there exists a minimal completely prime divisor of $P$. Moreover if $B$ is a completely prime ideal in $T$ such that $P \subset B$ then there exists a minimal completely prime divisor $M$ of $P$ such that $P \subset M \subset B$.

Theorem 3.15: If $P$ is a proper ideal in an $A$ – ternary semigroup $T$, then the radical of $P$ is the intersection of all minimal prime divisors of $P$.

Proof: Let $\{ P_i \}_{i \in I}$ denote the collection of all prime ideals in $T$ containing $P$. Let $\hat{I} = \{ i \in \hat{I} | \exists P_i \text{ is a minimal prime divisor of } P \}$. It must be shown that $\sqrt{P} = \bigcap_{i \in \hat{I}} P_i$. Let $y \in \sqrt{P}$.

Definition 3.5 implies $y \in \bigcap_{i \in \hat{I}} P_i$. Thus $y \in P_i$ for any $i \in I$ and it follows that $y \in P_i$ for any $i \in \hat{I}$. Therefore $y \in \bigcap_{i \in \hat{I}} P_i$. Conversely assume that $\bigcap_{i \in \hat{I}} P_i \not\subset \sqrt{P}$.
Ideal Theory in A-Ternary Semigroups

Then there exists an \( y \in \bigcap_{i=1} P_i \) such that \( y \notin \sqrt{P} \). Hence, there exists a prime ideal \( P_{i_0} \) in \( T \) such that \( y \in P_{i_0} \). Since \( P \subset P_{i_0} \) is prime, by theorem 3.14, there exists a minimal prime divisor \( P_j \in \{P_i\}_{i \in I}^* \) such that \( P \subset P_j \subset P_{i_0} \). However, \( P_j \in \{P_i\}_{i \in I}^* \) and \( y \in \bigcap_{i=1} P_i \) implies that \( y \in P_j \subset P_{i_0} \), a contradiction.

**Definition 3.16:** Let \( P \) be an ideal in a ternary semigroup \( T \). An element \( y \) in \( T \) will be called a \( A^- \) element for \( P \) if \( y = y_1 y_2 \ldots y_n \) such that \( y_1^2 y_2^2 \ldots y_n^2 \in P \) for some \( y_i \in T \) and for some odd positive integer \( n \).

The set of all \( A^- \) elements for the ideal \( P \) will be denoted by \( A^1(P) \). The ideal generated by \( A^1(P) \) will be denoted by \( A_1(P) \). If \( m > 1 \), then \( A_m(P) \) is defined recursively as follows:

\[
A_m(P) = A_1(A_{m-1}(P)).
\]

It is clear that \( A_m(P) \subset A_{m+1}(P) \) for each positive integer \( m \).

**Definition 3.17:** If \( P \) is an ideal in a ternary semigroup \( T \), then the Thierrin radical of \( P \) is denoted by \( A^*(P) \) and defined by

\[
A^*(P) = \bigcup_{m=1}^{\infty} A_m(P)
\]

**Theorem 3.18:** If \( P \) is an ideal in a ternary semigroup \( T \), the Thierrin radical of \( P \) is the intersection of all minimal completely prime divisors of \( P \).

The following theorem will show that the radical of a proper ideal in an \( A^- \) ternary semigroup is a specialization of the Thierrin radical.

**Theorem 3.19:** If \( P \) is a proper ideal in an \( A^- \) ternary semigroup \( T \), then \( \sqrt{P} = A^*(P) \)

**Proof:** Theorem 2.3 implies that the notion of prime ideal and completely prime ideal are equivalent in a commutative ternary semigroup. Thus, the collection of all minimal prime divisors of \( P \) is identical to the collection of all minimal completely prime divisors of \( P \). The Theorem follows from Theorem 3.15 and Theorem 3.18.

**Definition 3.20:** Let \( A \) be a proper ideal in a ternary semigroup \( T \). If \( x, y, z \in T \), \( xyz \in A \) and \( x \notin A \); \( y \notin A \) implies \( z^n \in A \) for some positive odd integer \( n \), then \( P \) is said to be primary.

The following theorem is an immediate consequence of theorem.

**Theorem 3.21:** If \( P \) is a proper ideal in an \( A^- \) ternary semigroup \( T \), then the following statements are equivalent.

- \( P \) is a primary.
- If \( x, y, z \in T \) such that \( x \notin P \), \( y \notin P \) and \( xyz \in P \) then \( z \in \sqrt{P} \).
- If \( x, y, z \in T \) such that \( xyz \in P \) and \( y, z \notin \sqrt{P} \) then \( x \notin P \).

**Theorem 3.22:** If \( P \) is a primary ideal in an \( A^- \) ternary semigroup \( T \), then \( \sqrt{P} \) is prime.

**Proof:** Let \( x, y, z \in T \) such that \( x \notin \sqrt{P} \), \( y \notin \sqrt{P} \) and \( xyz \in \sqrt{P} \). Since, \( xyz \in \sqrt{P} \), by theorem 3.9, there exists an \( n \in \mathbb{Z}^+ \) such that \((xyz)^n \in P \). Since \( T \) is commutative, it follows
that $x^ny^n z^n \in P$. Since $x, y \notin \sqrt{P}$, by theorem 3.9, $x^ny^n \notin \sqrt{P}$. Since $P$ is primary, $x^ny^n \notin P$ and $x^ny^n z^n \in P$. By theorem 3.21, we have $z^n \in \sqrt{P}$. Thus, there exists an $m \in Z^+$ such that $(z^n)^m \in P$. Clearly $P = nm \in z^+$ and $z^p = (z^n)^m \in P$. Therefore $Z \in \sqrt{P}$ and it follows that $\sqrt{P}$ is prime.

The following is an immediate consequence of theorem 3.22.

**Corollary 3.23:** If $P$ is a primary ideal in an $A$-ternary semigroup $T$, then $\sqrt{P}$ is the unique minimal prime divisor of $P$.

**Proof:** If $P$ is a primary ideal in an $A$-ternary semigroup $T$, theorem 3.22 implies that $\sqrt{P}$ is prime.

**Theorem 3.24:** Let $A_1, A_2, \ldots , A_n$ be primary ideals in an $A$-ternary semigroup $T$. If $\sqrt{A_i} = A$ for each $i = 1, 2, \ldots , n$, then $\bigcap_{i=1}^n A_i$ is primary and $\sqrt{\bigcap_{i=1}^n A_i} = \sqrt{A}$.

**Proof:** Let $y \in A$. Clearly $\sqrt{A_i} = A$ implies $y^{m_i} \in A_i$ for some $m_i \in N$, where $i = 1, 3 \ldots$.

Let $m = \max \{ m_i \}_{i=1}^n$. Thus $y^m = y^{m-i} y^{m_i}$ and $y^{m_i} \in A_i$ for each $i$. Thus, $y^m \in \bigcap_{i=1}^n A_i$.

Therefore $y \in \sqrt{\bigcap_{i=1}^n A_i}$.

Conversely, theorem implies that $\sqrt{\bigcap_{i=1}^n A_i} C \sqrt{A_i} = A$. Thus $\sqrt{\bigcap_{i=1}^n A_i} = A$. The following argument will show that $\bigcap_{i=1}^n A_i$ is primary. Let $a, b, c \in T$ such that $abc \in \bigcap_{i=1}^n A_i$ and $a, b, c \notin A$. Since each $A_i$ is primary, $abc \in A_i$ and $a, b, c \in A = \sqrt{A_i}$. It follows that $a \in A_i$ for each $i = 1, 2, \ldots , n$. Thus $a \in \bigcap_{i=1}^n A_i$.

**4. CONCLUSIONS**

In this paper we investigated the notion of an $A$-ternary semigroup and a characterization of an $A$-ternary semigroup was presented with the aid of these notions, further algebraic properties of the radical of an ideal in a $A$-ternary semigroup. Our future research work will concentrate on the applications of these systems ($A$-ternary semigroup) to civil engineering problems.

**REFERENCES**


Ideal Theory in A-Ternary Semigroups


