SEPARATION PROPERTIES IN IDEAL CLOSURE SPACES

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ABSTRACT

The intention of this paper is to characterize normal and regular separation axioms in ideal closure spaces.

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1 INTRODUCTION

Separation axioms constitute a classical topic in general topology. These axioms are statements about richness of topology. These axioms concern, the separation of points, point from closed sets and closed set from each other. Separation axioms in closure spaces have implication than corresponding axioms in topological spaces. According to E. Cech[2], a closure space is said to be separated if any two distinct points are separated by distinct neighbourhood. W.J. Thron[5] studied higher separation properties in closure spaces. The present paper aims at studying the higher separation axioms of ideal closure spaces. The main purpose of this paper to study the properties of regular spaces, normal spaces and completely normal spaces. we also have discuss a series of theorems and important results as well as basic concepts and the relationships of it.

2 PRELIMINARIES

In this section, we recall the basic definitions of ideal closure spaces.

Definition 2.1 [1] (X, ) be a topological space. An Ideal I on a topological space is a collection of non empty collections of subsets of X which satisfies:

(1)   I
(2) A  I, B  A implies B  I,
(3) $A \in I, B \in I$ implies $A \cup B \in I$.

If $(X, \mathcal{S})$ is a topological space and $I$ is an Ideal on $X$. Then $(X, \mathcal{S}, I)$ is called an Ideal topological space or an Ideal space.

**Definition 2.2** [10] Let $P(X)$ be the power set of $X$. Then the operator

$(*): P(X) \to P(X)$ is called a local function of $A$ with respect to $\mathcal{S}$ and $I$, is define as follows: For $A \subseteq X$, $A^*(I, \mathcal{S}^*) = \{x \in X : U \cap A \notin I \text{ for every open set } U \text{ containing } x \}$

Additionally, $\text{cl}^*(A) = A \cup A^*$ defines Kuratowski closure operator for a topology $\mathcal{S}^*$.

Here $\mathcal{S}^*$ is finer than $\mathcal{S}$.

**Definition 2.3** [7] Let $(X, k)$ be a non-empty set. $I$ be an Ideal on $X$.

Let $A^*: P(X) \to P(X)$ be a function of $A$ with respect to $I$ and $\mathcal{S}_{\mathcal{S}}$.

Let $k^*(A) = A \cup A^*$ defines Kuratowski closure operator for a topology.

Then the function $k^* = P(X) \to P(X)$ satisfying,

1. $k^*(\emptyset) = \emptyset$
2. $A \subseteq k^*(A)$ \quad $\forall A \subseteq X$
3. $k^*(A \cup B) = k^*(A) \cup k^*(B)$ \quad $\forall A, B \subseteq X$
4. $k^*(A) = k^*(k^*(A))$ \quad $\forall A \subseteq X$ is called a closure operator on $X$.

The structure $(X, I, k^*)$ is called an Ideal Closure Space.

**Example 2.4** $X = \{a, b, c\}$ \quad $\mathcal{S} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ \quad $I = \{\emptyset, \{c\}\}$

1. $A = \{a, c\}$ \quad $A^* = \{a, b\}$ \quad $k^*(A) = A \cup A^* \quad k^*\{a, c\} = X$.
2. $A = \{b, c\}$ \quad $A^* = \{b\}$ \quad $k^*(A) = A \cup A^* \quad k^*\{b, c\} = \{b, c\}$.
3. $A = \{a, b\}$ \quad $A^* = \{a, b\}$ \quad $k^*(A) = A \cup A^* \quad k^*\{a, b\} = \{a, b\}$.
4. $A = X$ \quad $A^* = \{a, b\}$ \quad $k^*(A) = A \cup A^* \quad k^*(X) = X$
5. $A = \emptyset$ \quad $A^* = \emptyset$ \quad $k^*(A) = A \cup A^* \quad k^*(\emptyset) = \emptyset$.
6. $A = \{a\}$ \quad $A^* = \{a, b\}$ \quad $k^*(A) = A \cup A^* \quad k^*\{a\} = \{a, b\}$.
7. $A = \{b\}$ \quad $A^* = \{b\}$ \quad $k^*(A) = A \cup A^* \quad k^*\{b\} = \{b\}$.
8. $A = \{c\}$ \quad $A^* = \emptyset$ \quad $k^*(A) = A \cup A^* \quad k^*\{c\} = \{c\}$.

Then $(X, I, k^*)$ is an Ideal Closure Space.

**Definition 2.5** [7] A subset $A$ of an Ideal closure space $(X, I, k^*)$ is said to be closed if $k^*(A) = A$.

**Definition 2.6** [7] A subset $A$ of an Ideal closure space $(X, I, k^*)$ is said to be open if $k^*(X - A) = X - A$ (i.e) $\text{Int}(A) = A$.

**Definition 2.7** [7] The set $\text{Int} A$ with respect to the closure operator $k^*$ is defined as $\text{Int}A = X - k^*(X - A)$ (i.e) $[k^*(A)^C]^C$, where $A^C = X - A$.

**Definition 2.8** [7] $(X, I, k^*)$ is an Ideal closure space than the associate topology on $X$ is $\mathcal{S}^* = \{A^C; k^*(A) = A\}$. Here $\mathcal{S}$ is not equal to $\mathcal{S}^*$.
Definition 2.9 [7] A subset A in an Ideal closure space \((X, I, k^*)\) is called neighbourhood of \(x\) if \(x \in \text{Int}(A)\).

Definition 2.10 [7] Let \((X, I, k^*)\) be an Ideal closure space. An Closure space \((Y, I, k^*_y)\) is called a subspace of \((X, I, k^*)\) if \(Y \subseteq X\) and \(k^*_y(A) = k^*(A) \cap Y, \forall A \subseteq Y\).

3 REGULAR AND \(T_3\)-SPACES

In this section, we introduced and characterise set of axioms by considering separation of closed sets using open sets.

Definition 3.1 An Ideal closure space \((X, I, k^*)\) is said to be regular if for each point \(x \in X\) and for each closed set \(F\) not containing \(x\), there exist an open set \(U_1\) and \(U_2\) such that \(F \subseteq U_2, x \in U_1\) and \(U_1 \cap U_2 = \emptyset\).

Definition 3.2 A regular \(T_1\) space is called a \(T_3\) space.

Example 3.3 \(X = \{a, b, c\}\), \(I = \{\emptyset, \{a, b\}\}\).

Ideal closure space \((X, I, k^*)\) is defined by \(k^*(a) = \{a\}; k^*(b) = \{b\}; k^*(c) = \{c\}; k^*\{a, b\} = \{a, b\}; k^*\{b, c\} = \{b, c\}; k^*\{a, c\} = \{a, c\}; k^*(X) = X; k^*(\emptyset) = \emptyset\)

Closed and open sets are \(X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\)

Let \(x = a, \ F = \{b, c\}, \ U_1 = \{a\}, \ U_2 = \{b, c\}\). Therefore \(U_1 \cap U_2 = \emptyset\)

Let \(x = b, \ F = \{a\}, \ U_1 = \{b, c\}, \ U_2 = \{a\}\). Therefore \(U_1 \cap U_2 = \emptyset\)

Let \(x = c, \ F = \{a, b\}, \ U_1 = \{c\}, \ U_2 = \{a, b\}\). Therefore \(U_1 \cap U_2 = \emptyset\)

Here \((X, I, k^*)\) is regular and every singleton set is closed in \((X, I, k^*)\). Therefore \((X, I, k^*)\) is \(T_3\)-space.

Theorem 3.4 Every \((X, I, k^*)\) is \(T_3\) - space than it is a \(T_2\) - space.

Proof: Let \((X, I, k^*)\) be a \(T_3\) - space. Let \(x, y \in X\) with \(x \neq y\). Since \((X, I, k^*)\) is \(T_3\) - space. \(\{y\}\) is closed \(x \neq y\). But \((X, I, k^*)\) is regular, by the definition of regular, there exists an open sets \(U_1, U_2\) such that \(F \subseteq U_2, x \in U_1\) and \(U_1 \cap U_2 = \emptyset\). Thus \(x \in U_1, \ y \in \{y\} \subseteq U_2\) and \(U_1 \cap U_2 = \emptyset\). Therefore \((X, I, k^*)\) is \(T_2\) - space.

Theorem 3.5 Every subspace \((Y, I, k^*_y)\) of ideal closure \(T_3\) - space is \(T_3\).

Proof: First we want to prove \((Y, I, k^*_y)\) is \(T_3\) space. If \(x, y \in Y\) such that \(x \neq y\). Since \((Y, I, k^*_y) \subseteq (X, I, k^*)\), then there exist open sets \(U_1\) and \(U_2\) such that \(x \in U_1, y \notin U_1\) and \(y \in U_2, x \notin U_2\). Therefore, \(x \in U_1 \cap Y\) and \(y \notin U_1 \cap Y\) and \(y \in U_2 \cap Y\), \(x \notin U_2 \cap Y\). Therefore \((Y, I, k^*_y)\) is \(T_3\) space.

Now we have to prove that \((Y, I, k^*_y)\) is regular space. Let \(y \in Y\) and \(G\) be a closed set in \(Y\) such that \(y \notin G\). Then \(G = Y \cap F\) for some closed set \(F\) in \(X\). Hence \(y \notin Y \cap F\), but \(y \in Y\) so \(y \notin F\). Since \((X, I, k^*)\) is \(T_3\) space so there exist open sets \(U_1\) and \(U_2\) in \((X, I, k^*)\) such that \(y \in U_1, F \subseteq U_2\) and \(U_1 \cap U_2 = \emptyset\). Take \(W = Y \cup U_1, Z = Y \cap U_2\) then \(W, Z\) are open sets in...
(Y, I, k_\*y) such that y \in W, G \subseteq Y \cap U_2 = Z and W \cap Z \subseteq U_1 \cap U_2 = \emptyset. Thus (Y, I, k_\*y) is T_3 space. □

**Theorem 3.6** Let (X, I,k*) be an ideal closure space. Then the following statements are equivalent.

(i) The space (X, I,k*) is regular.

(ii) For each point x \in X and for each open set U of x there exists an open set V of x such that k*(V) \subseteq U.

(iii) For each point x \in X and for each closed set F not containing x there exists an open set U_1 of x such that k*(U_1) \cap F = \emptyset.

**Proof:**

(i) ⇒ (ii)

Let (X, I,k*) be regular. Let x \in X and let U be any open set of (X, I,k*). Then X-U is closed subset of (X, I,k*). Since (X, I,k*) is regular, there exists an open neighbourhood U_1 of x and open neighbourhood U_2 of X-U such that U_1 \cap U_2 = \emptyset. Now U_1 \cap U_2 = \emptyset this implies U_1 \subseteq X - U_2, we get k*(U_1) \subseteq k*(X - U_2) = X - U_2, since X - U_2 is closed then we get k*(U_1) \cap U_2 = \emptyset, but k*(U_1) \cap X - U \subseteq k*(U_2) \cap U_2 = \emptyset. So k*(U_1) \cap X - U = \emptyset, then k*(V) \subseteq U.

(ii) ⇒ (iii)

Let x \in X and let F be a closed set in (X, I,k*) with x \notin F, X-F is open. So X-F is an open neighbourhood of x. Hence by (ii), there exist an open neighbourhood U_1 of x such that k*(U_1) \subseteq X - F. Therefore k*(U_1) \cap F = \emptyset.

(iii) ⇒ (i)

Let x \in X and let F be a closed set in (X, I,k*) with x \notin F, X-F is open. So X-F is an open neighbourhood of x. Hence by (iii), there exists an open neighbourhood U_1 of x such that k*(U_1) \cap F = \emptyset. Therefore, F \subseteq X - k*(U_1) = U_2_1, so U_2 is an open neighbourhood of F. U_1 \subseteq k*(U_1) \Rightarrow X - k*(U_1) \subseteq X - U_1. Then we get U_1 \cap (X - k*(U_1)) \subseteq U_1 \cap (X - U_1) = \emptyset. Therefore U_1 \cap U_2 = \emptyset. Hence (X, I,k*) is regular. □

**Theorem 3.7** Every subspace of regular Ideal closure space (X, I,k*) is regular.

**Proof:** Let (X, I,k*) be a regular ideal closure space. Let (Y, I,k*y) be a subspace of (X, I,k*). Let x \in X and V be a k_\*y -closed set in (Y, I,k*y). Therefore V = Y \cap L, where L is k_\* -closed in (X, I,k*). Since (X, I,k*) is regular, there exist k_\* -open sets U_1 and U_2 such that L \subseteq U_2, x \in U_1 and U_1 \cap U_2 = \emptyset. Therefore, there exist k_\*y -neighbourhood U_1 \cap Y of V and k_\*y -neighbourhood U_2 \cap Y of x such that (U_1 \cap Y) \cap (U_2 \cap Y) = \emptyset. Thus (Y, I,k*y) is also regular. □

**Theorem 3.8** If (X, I, \*3) is regular, then (X, I,k*) is also regular.

**Proof:** Let (X, I, \*3) be regular. Then for each point x \in X and each \*3-open sets U_1 and U_2 such that x \in U_2, F \subseteq U_1 and U_1 \cap U_2 = \emptyset. Since, neighbourhood in (X, I, \*3) is also a
neighbourhood of \((X, I, k^*)\), \(U_1\) and \(U_2\) are neighbourhood and \(F\) of \(x\) in \((X, I, k^*)\) respectively such that \(U_1 \cap U_2 = \emptyset\). Therefore \((X, I, k^*)\) is also regular. □

**Theorem 3.9** Let \((X, I, k^*)\) be an regular ideal closure spaces. If and only if for each \(x \in X\) and a closed set \(F\) in \((X, I, k^*)\) such that \(x \notin F\), there exist open sets \(U_1\) and \(U_2\) in \((X, I, k^*)\) such that \(x \in U_1\) and \(F \subseteq U_2\) and \(k^*(U_1) \cap k^*(U_2) = \emptyset\).

**Proof:** For each \(x \in X\) and a closed set \(F\) such that \(x \notin F\), by theorem 3.6(ii), there is an open set \(U_1\) such that \(x \in U_1\), \(k^*(U_1) \subseteq X \setminus F\). Since by theorem 3.6(ii), there is an open set \(U\) containing \(x\) such that \(k^*(U) \subseteq U_1\). Let \(U_2 = X \setminus k^*(U_1)\).

Then \(k^*(U) \subseteq U_1 \subseteq k^*(U_1) \subseteq X \setminus F\) implies \(F \subseteq X \setminus k^*(U_1) = U_2\) or \(F \subseteq U_2\). Also \(k^*(U_1) \cap k^*(U_2) = k^*(U) \cap k^*(X \setminus k^*(U_1)) \subseteq U_1 \cap k^*(X \setminus k^*(U_1)) \subseteq k^*(U_1) \cap k^*(X \setminus k^*(U_1)) = \emptyset\).

Thus \(U_1, U_2\) are the required open sets in \((X, I, k^*)\). This proves the necessity. The sufficiency is immediate. □

**Theorem 3.10** Let \((X, I, k^*)\) be an ideal closure space, \(F\) is a subset of \((X, I, k^*)\) and \(x \in X\). Then,

(i) \(x \in F\) if and only if \(\{x\} \subseteq F\).

(ii) If \(\{x\} \cap F = \emptyset\) then \(x \notin F\).

**Proof:** The proof is obvious. □

**Theorem 3.11** Let \((X, I, k^*)\) be an ideal closure space, \(x \in X\), if \((X, I, k^*)\) is regular space then,

(i) \(x \notin F\) if and only if \(\{x\} \cap F = \emptyset\) for every closed set \(F\).

(ii) \(x \notin G\) if and only if \(\{x\} \cap G = \emptyset\) for every open set \(G\).

**Proof:**

(i) Let \(F\) be a closed set such that \(x \notin F\). Since \((X, I, k^*)\) is regular space. Then by the definition of regular space there exists an open set \(G\) such that \(x \in G\) and \(G \cap F = \emptyset\). It follows that \(\{x\} \subseteq G\) from theorem 3.10 (i). Hence \(\{x\} \cap F = \emptyset\). Conversely, if \(\{x\} \cap F = \emptyset\), then \(x \notin F\) from theorem 3.10 (ii).

(ii) Let \(G\) be an open set such that \(x \notin G\). If we take \(\{x\} \cap G \neq \emptyset\). Hence, \(X \setminus G\) is closed set such that \(x \not\in X \setminus G\) it follows by (i) \(\{x\} \cap X \setminus G = \emptyset\). This implies that, \(\{x\} \subseteq G\) and so \(x \in G\). Which is contradiction to \(\{x\} \cap G = \emptyset\). Therefore \(\{x\} \cap G = \emptyset\). Conversely if \(\{x\} \cap G = \emptyset\) then it is obvious that \(x \notin G\). This completes the proof. □

**Corollary 3.12** Let \((X, I, k^*)\) be an ideal closure space and \(x \in X\). If \((X, I, k^*)\) is regular space then the following are equivalent.

(i) \((X, I, k^*)\) is \(T_1\)-space.

(ii) every \(x, y \in X\) such that \(x \neq y\), there exist open set \(U_1\) and \(U_2\) such that \(\{x\} \subseteq F\) and \(\{x\} \cap U_1 = \emptyset\) and \(\{y\} \subseteq U_2\) and \(\{x\} \cap U_2 = \emptyset\)

**Proof:** It is obvious from theorem 3.11 □

**Theorem 3.13** Let \((X, I, k^*)\) be an ideal closure space and \(x \in X\). Then the following are equivalent.

(i) \((X, I, k^*)\) is regular space.
(ii) For every closed set G such that \( \{x\} \cap G = \emptyset \) there exist open sets \( U_1 \) and \( U_2 \) such that \( \{x\} \subseteq U_1, \ G \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \).

**Proof:**

(i) \( \Rightarrow \) (ii)

Let \( G \) be a closed set such that \( \{x\} \cap G = \emptyset \) then \( x \notin G \) from theorem 3.12(i). It follows by (i), there exist open sets and \( U_1 \) and \( U_2 \) such that \( x \in U_1, \ G \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \).

(ii) \( \Rightarrow \) (i)

Let \( G \) be a closed set such that \( x \notin G \). Then \( \{x\} \cap G = \emptyset \) theorem 3.11. It follows by (ii), there exist open sets \( U_1 \) and \( U_2 \) such that \( \{x\} \subseteq U_1, \ G \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \). Hence \( x \in U_1, \ G \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \). Thus \( (X, I, k^*) \) is regular space.

4 NORMAL AND \( T_i \)-SPACES

**Definition 4.1** An Ideal closure space \( (X, I, k^*) \) is said to be normal, if for each pair of disjoint closed set \( F_i \) and \( F_j \) there exist disjoint open sets \( U_1 \) and \( U_2 \) such that \( F_i \subseteq U_1, \ F_j \subseteq U_2 \) such that \( U_1 \cap U_2 = \emptyset \).

**Example 4.2** \( X = \{a, b, c\}, \ \mathcal{I} = \{X, \emptyset, \{b\}, \{a\}, \{a, b\}\}, \ \mathcal{I} = \{\emptyset, \{a, b\}\} \).

**Ideal closure space** \( (X, I, k^*) \) is defined by \( k^*(a) = \{a, c\} ; \ k^*(b) = \{b, c\} ; \ k^*(c) = \{c\} ; \ k^*\{a, b\} = \{a, b\} ; \ k^*\{b, c\} = \{b, c\} ; \ k^*\{c, a\} = \{a, c\} ; \ k^*(X) = X ; \ k^*(\emptyset) = \emptyset \).

**Closed sets are** \( X, \emptyset, \{a, c\}, \{b, c\}, \{c, a\}, \{c\} \)

**Open sets are** \( X, \emptyset, \{a\}, \{b\}, \{c\} \)

Let disjoint closed set \( F_1 = \{a, b\}, F_2 = \{c\} \) then there exists disjoint open sets \( U_1 = \{a, b\}, \ U_2 = \{c\} \) such that \( F_1 \subseteq U_1, \ F_2 \subseteq U_2 \) Therefore \( U_1 \cap U_2 = \emptyset \).

Then \( (X, I, k^*) \) is normal space.

**Definition 4.3** A \( T_i \)-space is a normal \( T_i \)-space.

**Theorem 4.4** Every ideal closure space \( (X, I, k^*) \) \( T_i \)-space is also \( T_i \)-space.

**Proof:** Let \( (X, I, k^*) \) be a \( T_i \)-space then \( (X, I, k^*) \) is normal as well as \( T_i \)-space. To prove that the ideal closure space is \( T_i \)-space, it sufficient to show that the space is regular.

Let \( F \) be closed subset of \( (X, I, k^*) \) and let \( x \) be a point of \( X \) such that \( x \notin F \). Since \( (X, I, k^*) \) is a \( T_i \)-space. Thus \( \{x\} \) is closed subset of \( X \), such that \( \{x\} \cap F = \emptyset \), then by normality, there exist open sets disjoint \( U_1, U_2 \) such that \( \{x\} \subseteq U_1, \ F \subseteq U_2 \). Also \( \{x\} \subseteq U_1 \) then \( x \in U_1 \), then there exist open sets \( U_1, U_2 \) such that \( x \in U_1, \ F \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \). It follows that space \( (X, I, k^*) \) is regular space and also \( T_i \)-spaces.

**Theorem 4.5** If \( (X, I, \mathcal{I}^*) \) is normal, then \( (X, I, k^*) \) is also normal.

**Proof:** Let \( (X, I, \mathcal{I}^*) \) be normal, if for each pair of disjoint closed set \( F_1 \) and \( F_2 \), there exists disjoint \( \mathcal{I}^* \)-open sets \( U_1 \) and \( U_2 \) such that \( F_1 \subseteq U_1, \ F_2 \subseteq U_2 \), \( U_1 \cap U_2 = \emptyset \). Since neighbourhood in \( (X, I, \mathcal{I}^*) \) is also a neighbourhood of \( (X, I, k^*) \) respectively such that \( U_1 \cap U_2 = \emptyset \). Therefore \( (X, I, k^*) \) is also normal.
Theorem 4.6 In an Ideal closure space \((X, I, k^*)\), every closed subspace of a normal space is also normal.

Proof: Let \((X, I, k^*)\) be a normal ideal closure space. Let \((Y, I, k_y^*)\) be subspace of \((X, I, k^*)\). Let \(F_1, F_2\) are \(k_y^*\) closed subsets in \((Y, I, k_y^*)\). Then there exists closed set \(F_3, F_4\) in \((X, I, k^*)\) such that \(F_1 = Y \cap F_3\) and \(F_2 = Y \cap F_4\). Since \(Y\) is closed in \((X, I, k^*)\), therefore \(F_1, F_2\) are two disjoint closed set in \((X, I, k^*)\). Then \((X, I, k^*)\) is normal then there exists open sets \(U_1, U_2\) in \((X, I, k^*)\) such that \(F_1 \subseteq U_1, F_2 \subseteq U_2, U_1 \cap U_2 = \emptyset\). But then \(F_1 \subseteq U_1 \cap Y\) and \(F_2 \subseteq U_2 \cap Y\), where \(Y \cap U_1, Y \cap U_2\) are disjoint open subsets of \((Y, I, k_y^*)\). This proves that \((Y, I, k_y^*)\) is normal.

Theorem 4.7 An ideal closure space \((X, I, k^*)\) is normal implies that given closed set \(U\) and open set \(V\) such that \(U \subseteq V\), there exist open set \(L\) and closed set \(G\) such that \(U \subseteq L \subseteq G \subseteq V\).

Proof: Let \((X, I, k^*)\) be ideal closure normal space. Let \(U \subseteq V\), where \(U\) is closed set, \(V\) is open set. Therefore \(U \cap (X - V) = \emptyset\) and \(X - V\) is closed. Therefore, there exist open sets \(L\) and \(W\) such that \(U \subseteq L, X - V \subseteq W\) and \(L \cap W = \emptyset\). \(X - V \subseteq W \Rightarrow X - W \subseteq V\). \(L \cap W = \emptyset \Rightarrow L \subset X - W \subseteq V\). Also \(U \subseteq L \subseteq X - W \subseteq V\). If we take \(X - W = G\), then \(G\) is closed set. Thus we have, \(U \subseteq L \subseteq G \subseteq V\).

Theorem 4.8 Let \((X, I, k^*)\) be an Ideal closure space. Then the following statements are equivalent.

(i) \((X, I, k^*)\) is normal.

(ii) For each closed set \(F\) and for each open set \(U\) of \(F\) there exists an open set \(V\) of \(F\) such that \(k^*(V) \subseteq U\).

(iii) For each pair of disjoint closed set \(F_1\) and \(F_2\) in \((X, I, k^*)\) there exist an open set \(U\) of \(F_1\) such that \(k^*(U) \cap F_2 = \emptyset\).

Proof:

(i) \(\Rightarrow\) (ii)

Let \((X, I, k^*)\) be normal. Let \(F\) be a closed set and \(U\) any open set of \(F\). Now \(F\) and \(X - U\) are closed in \((X, I, k^*)\) and \(F \subseteq U\) implies that \(F \cap (X - U) = \emptyset\). Since \(X\) is normal there exist an open set \(V\) of \(F\) and an open set \(W\) of \(X - U\) such that \(V \cap W = \emptyset\). Now \(V \cap W = \emptyset \Rightarrow V \subset X - W \Rightarrow k^*(V) \subset k^*(X - W)\), since \(X - W\) is closed. \(k^*(V) \cap W = \emptyset\).

But \(k^*(V) \cap (X - U) \subset k^*(X - U) \cap W = \emptyset\). So \(k^*(V) \cap (X - U) = \emptyset\). Thus \(k^*(V) \subseteq U\).

(ii) \(\Rightarrow\) (iii)

Let \(F_1, F_2\) be disjoint closed sets in \((X, I, k^*)\). Since \(F_1 \cap F_2 = \emptyset\), we have \(F_1 \subseteq X - F_2\), where \(X - F_2\) is open. Hence \(X - F_2\) is open set of \(F_1\). Therefore by (ii) there exists an open set \(U\) of \(F_1\) such that \(k^*(U) \subset X - F_2\). So \(k^*(U) \cap F_2 = \emptyset\).

(iii) \(\Rightarrow\) (i)

Let \(F_1\) and \(F_2\) be disjoint closed set in \(X\). Then, by (iii), there exist an open set \(U_1\) of \(F_1\) such that \(k^*(U_1) \cap F_2 = \emptyset\). Now \(F_2\) and \(k^*(U)\) are disjoint closed set in \(X\). Hence by (iii) again, there exists an open set \(U_2\) of \(F_2\) such that \(k^*(U_2) \cap k^*(U_1) = \emptyset\). But \(\emptyset \subset U_1 \cap U_2 \subset k^*(U_1) \cap k^*(U_2) = \emptyset\). Therefore \(U_1 \cap U_2 = \emptyset\). Hence \(X\) is normal space.
5 COMPLETELY NORMAL SPACES

Definition 5.1 An ideal closure space \((X, I, k^*)\) is said to be completely normal if for each pair of disjoint closed sets \(F_1\) and \(F_2\) in \(X\), there exist open sets \(U_1, U_2\) such that \(F_1 \subseteq U_1, F_2 \subseteq U_2\) and \(k^*(U_1) \cap k^*(U_2) = \emptyset\).

Theorem 5.2 If \((X, I, k^*)\) is completely normal then it is normal.

Proof: If \((X, I, k^*)\) is completely normal ideal closure space. Let \(F_1\) and \(F_2\) are two disjoint closed sets. since \((X, I, k^*)\) is completely normal, then there exists open sets \(U_1, U_2\) such that \(F_1 \subseteq U_1, F_2 \subseteq U_2\) and \(k^*(U_1) \cap k^*(U_2) = \emptyset\). By the axiom of Ideal closure space, we have \(U_1 \cap U_2 = \emptyset\). Thus \((X, I, k^*)\) is normal.

Theorem 5.3 If \((X, I, k^*)\) is completely normal then every subspace is normal.

Proof: Let \((X, I, k^*)\) is completely normal and \((Y, I, k_y^*)\) be subspace. Let \(F_1\) and \(F_2\) are any two disjoint closed sets in \((Y, I, k_y^*)\). Then \(F_1 = F_3 \cap Y\) and \(F_2 = F_4 \cap Y\), where \(F_3, F_4\) are closed sets in \((X, I, k^*)\). Then there exist \(U_1\) and \(U_2\) open sets in \(X\) such that \(F_3 \subseteq U_1, F_4 \subseteq U_2\) and \(U_1 \cap U_2 = \emptyset\). This implies \(F_3 \cap Y \subseteq U_1 \cap Y\), \(F_4 \cap Y \subseteq U_2 \cap Y\) and \((U_1 \cap Y) \cap (U_2 \cap Y) = \emptyset\). Thus \((Y, I, k_y^*)\) is normal.

6 CONCLUSION

In this paper, we have redefined and explored separation axioms namely regular, \(T_3\), normal, \(T_4\) and completely normal spaces in Ideal closure spaces. In addition, we also discussed some important results and relationship of it.

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