VISCOELASTIC MODELING OF AORTIC EXCESSIVE ENLARGEMENT OF AN ARTERY

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ABSTRACT

In this study we have considered a problem on application of viscoelasticity model. A mathematical model is purposed to study biological tissue. We have applied damped mass-spring Voigt model and the behavior of an excessive localized enlargement of an artery is studied by varying the periodic blood force and amount of damping in the viscoelastic tissue. Also, the pressure inside the artery is assumed to vary linearly over space. We have considered Voigt model by assuming blood flow as a sinusoidal function and viscoelastic tissues as Navier-Stokes equation. The problem is solved with the help of Laplace Transforms by using boundary conditions. The phenomena of damping are examined and evaluated using MATLAB.

Keywords: Voigt model, Navier-Stokes Equation, Laplace Transforms, Resonance, Aortic.

1. INTRODUCTION

An excessive enlargement of an artery is a balloon-like bulge in an artery, or less frequently in a vein. Whether due to a medical condition, genetic predisposition, or trauma to an artery, the force of blood pushing against a weakened arterial wall can cause excessive enlargement of an artery. Excessive enlargement of an artery occur most often in the aorta (the main artery from the heart) and in the brain, although peripheral excessive enlargement of an artery can occur elsewhere in the body. Excessive enlargement of an artery in the aorta is classified as thoracic aortic excessive enlargement of an artery if located in the chest and are abdominal aortic excessive enlargements of an artery if located in the abdomen. Aortic
excessive enlargement of an artery is usually cylindrical in shape. These excessive
enlargements of an artery can grow large and rupture or cause dissection which is a split
along the layers of the arterial wall. Each year, most of the deaths are due to rupture of aortic
excessive enlargement of an artery.

Many excessive enlargement of an artery are due to fatty materials like cholesterol.
Arteries are made up of cells that contain elastin and collagen. Under small deformations,
elastin demines the elasticity of the vessel and under large deformations; the mechanical
properties are governed by the higher tensile strength collagen fibers. During the passage of
time, arteries become less rigid, which is a cause for hypertension. The tendency towards
rupture of an artery is explained with elasticity and viscoelasticity.

Concerning viscoelasticity, some models are purposed by the researchers as Migliavacca and
(2006) presented viscoelasticity modeling of the prostate region using vibro-elastography;
Wong et al. (2006) studied the theoretical modeling of micro-scale biological phenomena in
human coronary; Keith et al. (2011) presented an article on aortic stiffness. Recently, Kakar
et al. (2013) studied viscoelastic model for harmonic waves in non-homogeneous viscoelastic
filaments.

In this study, we have developed a mathematical model for the aortic excessive
enlargement of an artery. The exact solution is obtained by using an approximation of the
boundary condition at the excessive enlargement of an artery wall. We have also investigated
the influence of the excessive enlargement of an artery wall damping. The influence of
forcing function frequency is summarized numerically.

2. THE MATHEMATICAL MODEL

The purposed mathematical model under consideration for idealized aortic excessive
enlargement of an artery is composed of three sections:
• The blood within the excessive enlargement of an artery.
• The wall of the excessive enlargement of an artery.
• The bodily fluid that surrounds the excessive enlargement of an artery.

The geometry of the excessive enlargement of an artery is shown as a sphere in Fig. 1. Let
$u(x, t)$ represents the displacement of the bodily fluid w.r.t. space and time, with $x$ measured
from the exterior wall of the excessive enlargement of an artery and it is in the direction of $x$
i.e. in the direction of the spinal fluid. $u(0, t)$ represents displacement at the exterior face of
the excessive enlargement of an artery wall, and it is of greatest interest to us.

![Fig. 1: Excessive enlargement of an artery Section](image_url)
The flow of the bodily fluid can be described by one-dimensional Navier-Stokes equation which is derived from the Law of Conservation of Momentum

\[ \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial P}{\partial x} - \mu \frac{\partial^2 v}{\partial x^2} = B \]  

where \( \rho \) is unit density, \( v \) is velocity, \( P \) is pressure, and \( \mu \) is viscosity of the bodily fluid, and \( B \) is the body force. The first term \( \rho \frac{\partial v}{\partial t} \) in Eq. (1) denotes the rate of change of momentum; the term \( \rho v \frac{\partial v}{\partial x} \) is the time independent acceleration of a fluid with respect to space and it is neglected because it is assumed that the space derivative of the velocity is small in comparison to the time derivative. The third term \( \frac{\partial P}{\partial x} \) in Eq. (1) is the pressure gradient. The term \( \mu \frac{\partial^2 v}{\partial x^2} \) in Eq. (1) is known as the diffusion term it is dependent on the viscosity of the fluid. If we assume bodily fluid is in-viscid, then \( \mu \frac{\partial^2 v}{\partial x^2} \) term can also be neglected. Hence in the absence of body force, Eq. (1) reduces to

\[ \rho \frac{\partial v}{\partial t} + \frac{\partial P}{\partial x} = 0 \]  

The displacement of the bodily fluid is denoted by \( u = \int v dt \) as defined above. Then \( \frac{\partial u}{\partial t} = v(t) \) and \( \frac{\partial^2 u}{\partial t^2} = \frac{\partial v}{\partial t} \) In addition, we will assume that the bodily fluid is slightly compressible and obeys the Constitutive Law:

\[ P = -\rho c^2 \frac{\partial u}{\partial x} \]  

where \( c \) is the speed of sound through the bodily fluid. From Eq. (2) and Eq. (3), we get wave equation:

\[ \frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t) \]  

Since Eq. (4) has two derivatives in time and two derivatives in space. Therefore, there are two initial conditions and two boundary conditions for the problem.
2.1 Initial conditions
Assuming that the fluid is initially at rest, the initial conditions are:

\[ u(x, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = 0 \quad (5) \]

2.2 Boundary conditions
The problem has two boundary conditions
1. The boundary condition at \( x = L \), the plane wave will vanish at some long distance.

\[ \frac{\partial}{\partial t} u(L, t) = -c \frac{\partial}{\partial x} u(L, t) \quad (6) \]

Since an arterial wall is called viscoelastic because it has properties of both an elastic solid and a viscous liquid. For sake of convenience, in modeling, we have taken the arterial tissue in the form of model. In this model, spring and dashpot are arranged in parallel as shown in Fig. 2.

\[ F_{\text{blood}} = kX_0 \cos \omega t \]

\[ F_{\text{spring}} \]

\[ F_{\text{damping}} \]

Fig. 2: Rheological Model for Viscoelastic Excessive enlargement of an artery Wall

2. The boundary condition at \( x = 0 \), this condition is derived from a force balance equation. According to Newton’s Second Law, the rate of change in momentum must be equal to the total forces acting on the wall of the artery. Thus the total force will be the contributions of bodily fluid and from the spring with dashpot. The force due to bodily fluid is

\[ F_{\text{fluid}} = -Pa \quad (7a) \]

If we assume the bodily fluid is slightly compressible, then we can write the pressure in terms of displacement as

\[ F_{\text{fluid}} = \rho c^2 a \frac{\partial}{\partial x} u(0, t) \quad (7b) \]

where, \( \rho \) is the density of the bodily fluid, \( c \) is the speed of sound, and \( a \) is the cross sectional area. The damping force is
\[ F_{\text{damping}} = -\gamma \frac{\partial}{\partial t} u(0,t) \]  
where, \( \gamma \) is the damping constant.

The force from the spring will be given by Hooke’s Law, Therefore

\[ F_{\text{spring}} = -k(X_m - X_b) \]

where \( k \) is the spring constant, \( X_m \) is the displacement of the mass which arises due to the motion of the exterior face of the arterial wall \( u(0,t) \), and \( X_b \) represents the displacement of the internal wall of the artery because of blocking element which creates contraction of path and increase in pressure, it can be given by a sinusoidal function \( X_s \cos \omega t \), where \( \omega \) is the frequency of the periodic force from the blood pressure and \( X_s \) is the maximum displacement of the inner wall and. Hence, the boundary condition at \( x = 0 \) becomes

\[ m \frac{\partial^2}{\partial t^2} u(0,t) = \rho c^2 a \frac{\partial}{\partial x} u(0,t) - \gamma \frac{\partial}{\partial t} u(0,t) - ku(0,t) + kX_s \cos \omega t \]  

There are two solutions of Eq. (7)

1. Exact solution
2. Approximate solution

In both the cases, \( kX_s \cos \omega t \) is forcing function and its frequency is adjusted to simulate resonance. We will investigate whether this resonance is a contributing factor in excessive enlargement of an artery rupture. In summary, we will find an exact and an approximate solution of Eq. (4), Eq. (5), Eq. (6) and Eq. (7e). The exact solution is obtained by taking Laplace Transform of the bodily fluid pressure term and solved it at the boundary condition \( x = 0 \) which resulted in the bodily fluid pressure term being coupled with the damping force. As a result, the combined bodily fluid-damping term could never approach zero and resonance could not be produced. The approximate solution of the boundary condition can be obtained by linear approximation of the change in displacement due to the bodily fluid pressure term from \( x = 0 \) to \( x = L \). This assumption leads to group the bodily fluid pressure effect with the spring force. As a result of which the damping term could approach to zero, therefore an interesting resonance phenomenon is observed.

3. SOLUTION OF THE MODEL

We start with partial differential equation for the wave equation:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]  
\[ t \geq 0, \quad 0 < x < L \]

The Laplace Transforms of the displacement \( u(x,t) \) are
Taking the Laplace Transform of the wave Eq. (8a), we get
\[ s^2 U(x, s) - su(x, 0) - \frac{\partial}{\partial t} u(x, 0) = c^2 \frac{\partial^2}{\partial x^2} U(x, s) \] (9)

From Eq. (5) and Eq. (9), we have
\[ s^2 U(x, s) = c^2 \frac{\partial^2}{\partial x^2} U(x, s) \] (10)
The general solution to this equation is
\[ U(x, s) = c_1 \cosh \left( \frac{s}{c} x \right) + c_2 \sinh \left( \frac{s}{c} x \right) \] (11)
Taking the Laplace Transform of boundary condition Eq. (6), we get
\[ sU(L, s) - u(L, 0) = -c \frac{\partial}{\partial x} U(L, s) \] (12)
Substituting the general solution (11), the initial condition, and the derivative of \(U(x, s)\) with respect to \(x\) into Eq. (12), and evaluating all at \(L\), we get
\[ s \left( c_1 \cosh \left( \frac{s}{c} L \right) + c_2 \sinh \left( \frac{s}{c} L \right) \right) = -c \left( \frac{s}{c} c_1 \sinh \left( \frac{s}{c} L \right) + \frac{s}{c} c_2 \cosh \left( \frac{s}{c} L \right) \right) \]
which gives
\[ c_1 = -c_2 \]
Thus the general solution can be expressed
\[ U(x, s) = c_1 \cosh \left( \frac{s}{c} x \right) - c_1 \sinh \left( \frac{s}{c} x \right) \] (13)
Evaluating \(U(x, s)\) at the boundary condition \(x = 0\) gives \(U(0, s) = c_1\)
3.1 Exact solution
Rearranging the terms from Eq. (7.5), we have

\[-\rho c^2 \frac{\partial}{\partial x} u(0,t) + \gamma \frac{\partial}{\partial t} u(0,t) + ku(0,t) = kX_0 \cos \omega t\] (14)

Taking the Laplace Transform of this boundary condition (Boyce et al (2005)), we get

\[-\rho c^2 a \frac{\partial}{\partial x} U(0,s) + m \left( s^2 U(0,s) - su(0,0) - \frac{\partial}{\partial t} u(0,0) \right) + \gamma \left( sU(0,s) - u(0,0) \right) + kU(0,s) = kX_0 \frac{s}{s^2 + \omega^2} \] (15)

Simplifying for initial conditions and grouping like terms gives

\[-\rho c^2 a \frac{\partial}{\partial x} U(0,s) + (ms^2 + \gamma s + k)U(0,s) = kX_0 \frac{s}{s^2 + \omega^2} \] (16)

Taking the derivative \( \frac{\partial}{\partial x} U(x,s) \) of the general solution (13) and evaluating at \( x = 0 \) gives

\[-\rho c^2 a \left( \frac{s}{c} \cosh \left( \frac{s}{c} \right) - \frac{s}{c} \sinh \left( \frac{s}{c} \right) \right) \left[ \frac{s}{c} \cosh \left( \frac{s}{c} \right) - \frac{s}{c} \sinh \left( \frac{s}{c} \right) \right] + (ms^2 + \gamma s + k) \left( c_i \cosh \left( \frac{s}{c} \right) - c_i \sinh \left( \frac{s}{c} \right) \right) = kX_0 \frac{s}{s^2 + \omega^2} \]

which simplifies to

\[\rho c a \left( \frac{s}{c} \cosh \left( \frac{s}{c} \right) - \frac{s}{c} \sinh \left( \frac{s}{c} \right) \right) + (ms^2 + \gamma s + k) c_i = kX_0 \frac{s}{s^2 + \omega^2} \] (17)

Dividing both sides by \( m \) and rearranging terms yields

\[c_i \left( s^2 + \frac{\rho ca + \gamma}{m} \right) + \frac{k}{m} X_0 = \left( \frac{k}{m} X_0 \right) \frac{s}{s^2 + \omega^2} \] (18)

Let \( \gamma_0 = \rho ca + \gamma \) and \( \omega_0^2 = \frac{k}{m} \). Then for \( x = 0 \)

\[U(0,s) = c_i = \frac{\left( \frac{k}{m} X_0 \right) s}{s^2 + \left( \frac{\gamma_0}{m} \right) \left( s + \omega_0^2 \right) \left( s^2 + \omega^2 \right)} \] (19)

Taking the inverse Laplace Transform of \( U(0,s) \) and using partial fraction decomposition

\[L^{-1}\left(U(0,s)\right) = L^{-1}\left( \frac{A}{s - r_1} + \frac{B}{s - r_2} + \frac{Cs}{s^2 + \omega_0^2} + \frac{D\omega}{s^2 + \omega_0^2} \right) \]

yields a solution of form
\[ u(0, t) = Ae^{\gamma_0} + Be^{\gamma_0} + C \cos \omega_0 t + D \sin \omega_0 t \]  \hspace{1cm} (20)

where:

\[ r_{i, i} = \frac{-\gamma_0 \pm \sqrt{\gamma_0^2 - 4m^2 \omega_0^2}}{2m} \quad \gamma_0 = \rho c\alpha + \gamma \quad \omega_0^2 = \frac{k}{m} \]

\[ A = \frac{(k/m) X r_i}{(r_i - r_j)(r_i^2 + \omega^2)} \quad B = \frac{-(k/m) X r_i}{(r_i - r_j)(r_i^2 + \omega^2)} \]

\[ C = \frac{(k/m) X (\omega_j^2 - \omega_i^2)}{(r_i^2 + \omega_i^2)(r_i^2 + \omega_i^2)} \quad D = \frac{(k/m^2) X \gamma_0 \omega}{(r_i^2 + \omega_i^2)(r_i^2 + \omega_i^2)} \]  \hspace{1cm} (21)

3.2 Approximation of the Boundary Condition at the Excessive enlargement of an artery Wall

In order to solve Eq. (7.5), we use the standard linear approximation for \( u(x + h, t) \) (Boyce et al. (2005)):

\[ u(x + h, t) = u(x, t) + h \frac{\partial}{\partial x} u(x, t) \]  \hspace{1cm} (22)

At \( h = L \), some distance far away from the arterial wall, \( u(L, t) = u(0, t) + L \frac{\partial}{\partial x} u(x, t) \).

Since \( u(L, t) = 0 \), we can solve for \( \frac{\partial}{\partial x} u(x, t) \), and substitute the expression \( \frac{\partial}{\partial x} u(x, t) = -\frac{u(0, t)}{L} \) into the bodily fluid pressure term in the boundary condition (7.5), giving:

\[ m \frac{\partial^2}{\partial t^2} u(0, t) + \gamma \frac{\partial}{\partial t} u(0, t) + \left( k + \frac{\rho c^2 a}{L} \right) u(0, t) = kX_0 \cos \omega_0 t \]  \hspace{1cm} (23)

Let \( k = k + \frac{\rho c^2 a}{L} \) and \( \omega_0^2 = \frac{k}{m} \)

Taking the Laplace Transform of this boundary condition (23) yields

\[ m \left[ s^2 U(0, s) - su(0, 0) - \frac{\partial}{\partial t} u(0, 0) \right] + \gamma \left[ sU(0, s) - u(0, 0) \right] + \tilde{k} U(0, s) = kX_0 \frac{s}{s^2 + \omega_0^2} \]  \hspace{1cm} (24)

Simplifying for initial conditions and grouping like terms gives:

\[ (ms^2 + \gamma s + \tilde{k}) U(0, s) = kX_0 \frac{s}{s^2 + \omega_0^2} \]  \hspace{1cm} (25)
Dividing both sides by \( m \) and solving:

\[
U(0,s) = \frac{(k/m) X_s s}{s^2 + \left(\frac{\gamma}{m}\right)s + \omega^2} (s^2 + \omega^2)
\]

Taking the inverse Laplace Transform of \( U(0,s) \) and using partial fraction decomposition

\[
L^{-1}(U(0,s)) = L^{-1} \left( \frac{A}{s-r_i} + \frac{B}{s-r_j} + \frac{Cs}{s^2 + \omega^2} + \frac{D\omega}{s^2 + \omega^2} \right)
\]

yields a solution of the same form as the exact solution Eq. (20)

i.e. \( u(0,t) = Ae^{\alpha t} + Be^{\beta t} + C \cos \omega t + D \sin \omega t \)

where:

\[
r_{ij} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4m^2\omega^2}}{2m} \quad \tilde{k} = k + \frac{\rho c^2 a}{L} \quad \omega_0 = \frac{k}{m}
\]

\[
A = \frac{(k/m) X_{r_i}}{(r_i - r_j)(r_i^2 + \omega^2)} \quad B = \frac{-(k/m) X_{r_j}}{(r_i - r_j)(r_j^2 + \omega^2)}
\]

\[
C = \frac{(k/m) X_{r_i} (\omega_i^2 - \omega^2)}{(r_i^2 + \omega^2)(\omega_i^2 + \omega^2)} \quad D = \frac{(k/m^2) X_{r_j} \gamma \omega}{(r_i^2 + \omega^2)(r_j^2 + \omega^2)}
\]

Discussion

Although the two solution processes are quite different but exact and approximate solutions appear very similar. The approximate solution is a solution corresponds to one boundary condition whereas the exact solution is a solution to the whole system. In exact solution, we obtain the term \( \gamma_0 = \rho ca + \gamma \) and in approximate solution, we get \( \tilde{k} = k + \frac{\rho c^2 a}{L} \) term (Boyce et al. 2005). \( r_i \) and \( r_j \) are the roots of the corresponding homogeneous equation. The effect of these terms vanishes quickly, it is the transient solution. The trigonometric terms contribute the particular solution of the non-homogeneous equation and their effect remains as long as the external force is applied. The steady-state solution is: (appendix)

\[
U(t) = R \cos(\omega t - \delta)
\]

where,

\[
R = \frac{kX_0}{\sqrt{m^2(\omega_i^2 - \omega^2)^2 + \gamma^2 \omega^4}}
\]

\[
\tan \delta = \frac{\gamma \omega}{m(\omega_0^2 - \omega^2)}
\]
4. NUMERICAL ANALYSIS

Various graphs are plotted for two solutions of the boundary condition at the wall. In these graphs, we have studied the effect of excessive enlargement of an artery wall damping and the effect of the frequency of forcing function due to blood pressure by choosing following parameters purposed by Salcudean et al. (2005).

Table 1

<table>
<thead>
<tr>
<th>ρ (kg/m³)</th>
<th>c (m/s)</th>
<th>a (m²)</th>
<th>γ (kg/s)</th>
<th>k (N/m)</th>
<th>m (kg)</th>
<th>X₀ (m)</th>
<th>ω (rad/s)</th>
<th>L (m)</th>
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</thead>
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<td>1000</td>
<td>1500</td>
<td>0.0001</td>
<td>2</td>
<td>8000</td>
<td>0.0001</td>
<td>0.002</td>
<td>6</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The Influence of Wall Damping and the influence of frequency are plotted for both exact and approximate solutions. It is observed that as the damping constant of the excessive enlargement of an artery wall increases, the maximum displacement decreases (Fig. 3). The frequency made small effect on the maximum displacement (Fig. 4). With the increase in frequency of the forcing function, the time period will decrease and the maximum displacement will also decrease. The value of \( \gamma = 200 \) is used in the exact solution but in for approximation, it is required a much smaller \( \gamma \) i.e. equal to 0.0002. Fig. 5 and Fig. 6 are plotted to explain the approximate solutions. The concept of resonance and beats are also shown in Fig. 7 and Fig. 8. It is observed that the transient solution dies away immediately in all of the graphs.

Phenomenon of resonance

Resonance occurs when \( \omega = \omegaₐ \). The maximum displacement is \( R = \frac{kX₀}{\sqrt{m^2(\omegaₐ^2 - \omega^2)^2 + \gamma^2 \omega^2}} \), then as \( \omega \to \infty \), \( R \to 0 \). Also, as \( \omega \to 0 \), the forcing amplitude \( R \to X₀ \). when \( \omega = \omega_{max} \), where \( \omega_{max} = \omegaₐ^2 - \frac{\gamma^2}{2m^2} \). The corresponding maximum displacement is \( R_{max} = \frac{kX₀}{\gamma \omegaₐ \sqrt{1 - (\frac{\gamma^2}{4mk})}} \). As the damping constant \( \gamma \to 0 \), the frequency of the maximum \( R \to \omegaₐ \) and the amplitude \( R \) increases without bound. i.e. in case of lightly damped systems, the amplitude of the forced vibrations becomes quite large even for small external forces. This phenomenon is known as resonance (Boyce et al. (2005)). (Fig. 8).

Phenomenon of beats

Beats occur when \( \omega \neq \omegaₐ \), in this case the amplitude varies slowly in a sinusoidal manner but the function oscillates rapidly. This periodic motion is called a beat (Fig. 7).
Fig. 3: Effect of Wall Damping \( (\omega = 2\pi) \) for Exact Solution

Fig. 4: Effect of Frequency \( (\gamma = 2) \) for Exact Solution
Fig. 5: Effect of Wall Damping ($\omega = 2\pi$) for Approximation of Boundary Condition

Fig. 6: Approximate Solution – Effect of Wall Damping at High Frequency ($\omega = \omega_x$)
Fig. 7: Beats

Fig. 8: Normalized amplitude of steady-state response v/s frequency of driving force
5. Conclusion

With the help of several assumptions, a complete solution of an excessive enlargement of an artery is obtained. A Voigt model is taken to model the excessive enlargement of an artery wall. It is observed that as the damping constant of the excessive enlargement of an artery wall increases, the maximum displacement decreases. With the increase in frequency of the forcing function, the time period decreases and the maximum displacement also decreases.

Appendix

1. The amplitude and phase shift of the sum of two sinusoids functions of equal frequency can be determined as:

Let \( f(t) = A \cos \alpha t \) and \( g(t) = B \sin \alpha t \) be two sinusoidal functions with the same period and they can be rewritten as a single sinusoid \( s(t) = R \cos(\alpha t - \delta) \) as follows:

\[
A \cos \alpha t + B \sin \alpha t = R \cos(\alpha t - \delta)
\]

\[
\Rightarrow A \cos \alpha t + B \sin \alpha t = R \cos \alpha t \cos \delta + R \sin \alpha t \sin \delta
\]

Let \( A = R \cos \delta \) and \( B = R \sin \delta \)

If we divide \( \frac{B}{A} \), we get \( \frac{B}{A} = \frac{R \sin \delta}{R \cos \delta} = \tan \delta \)

Consequently, the phase shift is

\[
\delta = \tan^{-1} \left( \frac{B}{A} \right).
\]

Also, if we squares \( A \) and \( B \) and adding, we get

\[
A^2 + B^2 = R^2 \cos^2 \delta + R^2 \sin^2 \delta
\]

\[
A^2 + B^2 = R^2(\cos^2 \delta + \sin^2 \delta)
\]

\[
\Rightarrow A^2 + B^2 = R^2
\]

Thus \( R = \sqrt{A^2 + B^2} \).

2. Sum and product of quadratic roots:

In checking over our solutions, we wanted to confirm that \( R = \sqrt{C^2 + D^2} \) where

\[
C = \frac{(k/m) X_s \left( \omega_s^2 - \omega^2 \right)}{(r_s^2 + \omega^2)(r_s^2 + \omega^2)} \quad \text{and} \quad D = \frac{(k/m) X_s \nu \omega}{(r_s^2 + \omega^2)(r_s^2 + \omega^2)}
\]

were the coefficients of the
trigonometric terms of the solution and \( R = \frac{k \gamma X_s}{\sqrt{m^2 (\omega_s^2 - \omega^2)^2 + \gamma^2 \omega^2}} \). We used the roots

\[
r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4m^2 \omega_s^2}}{2m}
\]

and \( \omega_s^2 = \frac{k}{m} \), and generated some very typical algebraic equations. The sum and product of quadratic roots are found. For the characteristic equation

\[
r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0 ; \quad a = \frac{1}{m}, \quad b = \frac{\gamma}{m} \quad \text{and} \quad c = \frac{k}{m}.
\]

Therefore, \( r_1 + r_2 = -\frac{b}{a} \) or \( -\frac{\gamma}{m} \) and \( r_1 \cdot r_2 = \frac{c}{a} \) or \( \frac{k}{m} \). These two facts are used to simplify the work.

REFERENCES