MATHEMATICAL MODEL OF CREEP BEHAVIOR IN AN ANISOTROPIC ROTATING DISC OF AL-SICP WITH THICKNESS VARIATION IN PRESENCE OF THERMAL RESIDUAL STRESS

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ABSTRACT

Creep response in a rotating disc made of anisotropic Al-SiC(particle) composite having linearly varying thickness has been carried out using anisotropic Hoffman yield criterion in present study. The steady state creep behavior is described by Sherby’s constitutive model. The material parameters characterizing difference in yield stresses have been used from the available experimental results in literature. In the presence of residual stress, stress and strain rate distributions got affected for isotropic composites with characteristic parameters $\alpha=1$ and anisotropic composites with characteristic parameters $\alpha=0.7$ and $1.3$, indicating strengthening or weakening in the tangential direction in composite disc presumably introduced by either processing or inhomogeneous distribution of reinforcement. It is concluded that the change in the stresses by anisotropy in composite disc with residual stress is relatively small while anisotropy introduces significant change in the strain rates.

Keywords: Composites, Steady state creep, Residual stress, Anisotropy.

1. INTRODUCTION

Stresses induced due to thermal mismatch between the metal matrix and the ceramic reinforcement in metal matrix composite may impart plastic deformation to the matrix thereby resulting in a reduction of the residual stresses. Thermal mismatch strains also may quite often crack the matrix resulting in a relaxation of the residual stresses. Presence of thermal residual stresses can induce the asymmetry in the tensile and compressive yield stresses of the composite. Because of its influence on the properties, the residual stress in composites has been the subject of several studies, both experimentally and analytically. Arya and Bhatnagar [2] have carried out creep analysis of orthotropic rotating disc using constitutive equations and a time hardening law. It is reported that the tangential stress at any
radius and the tangential strain rate at the inner radius decrease with increasing anisotropy of the material. Arsenault and Taya [3] investigated the magnitude of the thermal residual stresses by determining the difference of the yield stresses between tension and compression resulting from the thermal residual stresses. Mishra and Pandey[4] is claimed that Sherby’s constitutive creep model works better than Norton’s creep law to describe the creep behavior of aluminum matrix composites. Jahed and Shirazi [5] investigated loading and residual stresses, and associated strains and displacements in thermoplastic rotating discs at elevated temperatures. Singh and Ray [6] proposed a new yield criterion for residual stress, which at appropriate limits reduces to Hill anisotropic and Hoffman anisotropic yield criterion and carried out analysis of steady state creep in a rotating disc made of Al-SiCw composite using this criterion and compared the results obtained using Hill anisotropic yield criterion ignoring difference in yield stresses. The presence of tensile residual stress in the disc is observed to increase the creep rate significantly compared to that in a similar anisotropic disc without residual stress. Gupta et al. [7] have analyzed steady state creep in isotropic aluminum silicon carbide particulate rotating disc. The creep behavior has been described by Sherby’s law. It is observed that the tangential as well as radial stress distribution in the disc do not vary significantly for various combinations of material parameters and operating temperatures. The tangential as well as radial strain rates in the disc reduces significantly with reducing particle size, increasing particle content and decreasing operating temperature. Baykara [8] explained the stress concentration factors at a small circular hole in a large sheet of plastic anisotropy, subjected to equi-biaxial tension. In this analysis an orthotropic yield function suggested by Hill and approximate solution technique developed by Durban are used. Numerical results for aluminum and copper sheets are presented. Singh [9] have studied creep behavior of a whisker reinforced anisotropic rotating disc with constant thickness. He described creep behavior by Norton’s power law and concluded that anisotropy appears to help in restraining creep response both in the tangential and in the radial directions over the entire disc. Singh and Rattan [10] has investigated the stress distributions and the resulting creep deformation in isotropic rotating disc having constant thickness and made of silicon carbide particulate reinforced aluminum base composite in presence of thermal residual stress. Vandana and Singh [11] have studied the effect of anisotropy on the stress and strain rates in composite disc made of anisotropic material (6061Al - 30% vol SiCp) and concluded that the anisotropy of the material has a significant effect on the creep of a rotating disc with varying thickness.  

In this paper, the steady state creep has been analyzed in composite rotating disc made of material 6061Al base alloy containing 20vol% of SiC(particle). The thickness of disc is varying linearly and the creep behavior has been described by Sherby’s constitutive model. The effect of anisotropy has been investigated in terms of a single parameter indicating strengthening or weakening in the tangential direction in the disc with thermal residual stress.  

2. Mathematical formulation 

Consider a thin orthotropic composite disc of 6061 Al-SiCp of density $\rho$ and rotating at a constant angular speed $\omega$ radian/sec. The thickness of the disc is assumed to be $h$ and $a$ and $b$ be inner and outer radii of the disc respectively. Let $I$ and $I_0$ be the moment of inertia of the disc at inner radius $a$ and outer radius $b$ respectively. $A$ and $A_0$ be the area of cross section of disc at inner radius $a$ and outer radius $r$ and $b$ respectively. Then
\begin{align*}
I &= \int_{a}^{b} r^2 \, dr, \quad I_0 = \int_{a}^{b} h \, r^2 \, dr \\
A &= \int_{a}^{b} h \, dr, \quad A_0 = \int_{a}^{b} h \, dr \tag{1}\text{ and } \sigma_{\theta_{avg}} = \frac{1}{A_0} \int_{a}^{b} h \, \sigma_{\theta} \, dr. \tag{2}
\end{align*}

For the purpose of analysis of the disc the following assumptions are made:

1. Material of disc is orthotropic and incompressible
2. Elastic deformations are small for the disc and therefore they can be can be neglected as compared to creep deformation.
3. Axial stress in the disc may be assumed to be zero as thickness of disc is assumed to be very small compared to its diameter.
4. The composite shows a steady state creep behavior, which may be described by following Sherby’s law \([1]\),

\begin{align*}
\dot{\varepsilon} &= (M \left( \sigma - \sigma_0 \right))^{n} \tag{3}
\end{align*}

where \( \dot{\varepsilon}, \bar{\sigma}, n, \sigma_0, A, D, \lambda, b, E \) be the effective strain rate, effective stress, the stress exponent, threshold stress, a constant, lattice diffusivity, the sub grain size, the magnitude of burgers vector, Young’s modulus. The values of creep parameters \( m \) and \( \sigma_0 \) are described by the following regression equations as a function of particle size, temperature and volume percent, which extracted from the available experimental results of Pandey et al.\([6]\).

\( m = e^{-35.38} \ P^{0.2077} \ T^{4.98} \ V^{-0.622} \) \text{ and } \sigma_0 = -0.03507P + 0.01057T + 1.00536V - 2.11916

Taking reference frame along the principal directions of \( r, \theta \text{ and } z \), the generalized constitutive equations for an anisotropic disc under multiaxial stress condition are given as,

\begin{align*}
\dot{\varepsilon}_r &= \frac{\dot{\varepsilon}}{2\bar{\sigma}} \left\{ (G + H)\sigma_r - H\sigma_\theta - G\sigma_z \right\} \tag{4} \text{ and } \dot{\varepsilon}_\theta &= \frac{\dot{\varepsilon}}{2\bar{\sigma}} \left\{ (H + F)\sigma_\theta - F\sigma_z - H\sigma_r \right\} \\
\dot{\varepsilon}_z &= \frac{\dot{\varepsilon}}{2\bar{\sigma}} \left\{ (F + G)\sigma_z - G\sigma_r - F\sigma_\theta \right\} \tag{5}
\end{align*}

where \( F, G \text{ and } H \) are anisotropic constants of the material \( \dot{\varepsilon}_r, \dot{\varepsilon}_\theta, \dot{\varepsilon}_z \) and \( \sigma_r, \sigma_\theta, \sigma_z \) are the strain rates and the stresses respectively in the direction \( r, \theta \text{ and } z \). \( \bar{\varepsilon} \) be the effective strain rate, \( \bar{\sigma} \) be the effective stress. For biaxial state of stress \( (\sigma_r, \sigma_\theta) \) the effective stress is,

\begin{align*}
\bar{\sigma} &= \left\{ \frac{1}{(G+H)} \left[ F\sigma_\theta^2 + G\sigma_z^2 + H(\sigma_r - \sigma_\theta)^2 \right] \right\}^{1/2} \tag{7}
\end{align*}

Using Eqs.(3)and(7), Eq.(4) can be rewritten as,

\begin{align*}
\dot{\varepsilon}_r &= \frac{du_r}{dr} = \frac{\sqrt{H(F+G)\left[ \frac{1}{(G+H)} + \frac{G}{H} \right] x - 1 + \frac{f - \frac{f}{\sigma_\theta}}{\sigma_\theta} \left[ M(\bar{\sigma} - \sigma_0) \right]^n}}{2\left[ \left( 1 + \frac{G}{H} \right) x^2 - 2x + \left( 1 + \frac{F}{H} \right)^{n/2} \right]} \tag{8}
\end{align*}

where, \( x(r) = \frac{\sigma_r}{\sigma_\theta} \)

Similarly from Eq.(5),
\[ \dot{\varepsilon}_a = \frac{\dot{u}}{r} = \sqrt{H(F+G)\left[\left(1+\frac{F}{H}\right)x + \frac{f_x - f_r}{\sigma_0}\right]} \left( M(\sigma - \sigma_0) \right) \]

(9) and \[ \dot{\varepsilon}_z = -(\dot{\varepsilon}_r + \dot{\varepsilon}_\theta) \] (10)

Dividing (8) by (9),

\[ \phi(r) = \frac{G + H}{F} \left( \frac{x - \frac{H}{F} + \frac{f_x - f_r}{\sigma_0}}{1 + \frac{H}{F}} \right) + \frac{H x + \frac{f_x - f_r}{\sigma_0}}{1 + \frac{H}{F}} \]

(11)

The equation of equilibrium for a rotating disc with varying thickness can be written as,

\[ \frac{d}{dr}(r h \sigma_r - h \sigma_\theta + \rho \omega^2 r^2 h) = 0 \]

(12)

Integrating Eq.(12) within limits \(a\) to \(b\) and using Eqs.(1) and (2),

\[ \sigma_{\theta u} = \frac{1}{A_0} \rho \omega^2 I_0 \]

(13)

Using Eq.(8) and (9), Eq.(12) becomes,

\[ \sigma_\theta = \frac{\psi_1(r) \left[ \rho \omega^2 I_0 - \int_a^b \psi_2(r) \cdot h dr \right]}{\int_a^b \psi_1(r) \cdot h dr} + \psi_2(r) \]

(14)

where,

\[ \psi_1(r) = \left\{ \frac{F}{G + H} \left[ \frac{G}{F} \right] x^2 - \frac{2 H}{F} x + \left(1 + \frac{H}{F}\right) \right\}^{1/2} \]

(15) and \[ \psi_2(r) = \left\{ \frac{F}{G + H} \left[ \frac{G}{F} \right] x^2 - \frac{2 H}{F} x + \left(1 + \frac{H}{F}\right) \right\}^{1/2} \]

(16)

\[ \psi(r) = \left\{ \frac{2}{r} \left[ \frac{G + H}{F} \right] x^2 - \frac{H}{F} x + \left(1 + \frac{H}{F}\right) \right\}^{1/2} \]

(17)

Integrating Eq.(12) within limits \(a\) to \(r\) and use Eq.(1),

\[ \sigma_r = \frac{1}{r \cdot h} \left[ \int \sigma_\theta \cdot h dr - \rho \omega^2 I \right] \]

(18)

Thus the tangential stress \(\sigma_\theta\) and radial stress \(\sigma_r\) are determined by Eqs.(14) and (18). Then strain rates \(\dot{\varepsilon}_r, \dot{\varepsilon}_\theta\) and \(\dot{\varepsilon}_z\) calculated from Eqs.(4), (5) and (6).

**Tensile strength and Hill anisotropy constants**

When a sample disc material is tested under uniaxial loading in the \(r\) and \(\theta\) direction, the corresponding stress invariant may be written in term of the observed tensile strengths and the Hill anisotropy constants as given below:

\[ \sigma_i = \sqrt{\frac{G + H}{2} \sigma_{r y}} \]

(23) and \[ \sigma_i = \sqrt{\frac{F + H}{2} \sigma_{\theta y}} \]

(24)

where \(\sigma_i\) is the isotropic yield stress for which \(F/G = G/H = H/F = 1\)
When the sample is tested under uniaxial loading in the direction, the stress invariant may be written similarly as

$$\sigma_i = \sqrt{\frac{F + G}{2}} \sigma_{zy}$$

(25)

where $\sigma_{ry}$, $\sigma_{\theta y}$ and $\sigma_{zy}$ are the yield stresses in the $r$, $\theta$ and $z$ directions respectively.

Following Hill (1948), the anisotropy constants can be written in terms of these yield stresses as

$$F = \left( \frac{1}{\sigma_{\theta y}^2} + \frac{1}{\sigma_{zy}^2} - \frac{1}{\sigma_{ry}^2} \right) \sigma_i^2$$

$$G = \left( \frac{1}{\sigma_{zy}^2} + \frac{1}{\sigma_{ry}^2} - \frac{1}{\sigma_{\theta y}^2} \right) \sigma_i^2$$

and

$$H = \left( \frac{1}{\sigma_{ry}^2} + \frac{1}{\sigma_{\theta y}^2} - \frac{1}{\sigma_{zy}^2} \right) \sigma_i^2$$

For simplicity one may assume that the same isotropic yield stresses are observed in $r$ and $z$ directions and one may put $\sigma_{ry} = \sigma_{zy} = \sigma_i$ in Eqs. (23) and (25) to have $F/G = 1$

One may assume that a small anisotropy exists in the disc due to processing which is reflected in the difference in strengths circumferential and radial directions as characterized by the ratio of their respective yield strengths, $\alpha$, as given below $\sigma_{ry}/\sigma_{\theta y} = \alpha$

From Eqs. (23) and (24)

$$\alpha = \sqrt{\frac{F/H + 1}{G/H + 1}} = \sqrt{\frac{2}{G/H + 1}}$$

or

$$G = \frac{2}{H \alpha^2} - 1$$

The isotropic material corresponds to $\alpha = 1$ and the effect of a deviation of $\alpha$ from one on the resulting stress and creep will be investigated in the present study.

3. SOLUTION PROCEDURE

The stress distribution is evaluated from the above analysis by iterative numerical scheme of computation. In the first iteration, it is assumed that $\sigma_{\theta} = \sigma_{\theta_{avg}}$ over the entire disc radii.

Substituting $\sigma_{\theta_{avg}}$ for $\sigma_{\theta}$ in Eq. (18) the first approximation value of $\sigma_i$, i.e. $[\sigma_i]_1$ is obtained.

The first approximation of stress ratio, i.e. $[x]_1$, is obtained by dividing $[\sigma_i]_1$ by $\sigma_{\theta}$ which can be substituted in Eq. (11) to calculate first approximation of $\phi(r)$ i.e. $[\phi(r)]_1$. Now one carries out the numerical integration of $[\phi(r)]_1$ from limits of $a$ to $r$ and uses this value in Eq.(17) to obtain first approximation of $\psi(r)$ i.e. $[\psi(r)]_1$. Using this $[\psi(r)]_1$, in Eq. (15) and (16) respectively, $[\psi_i(r)]_1$, are found, which are used in Eq. (14) to find second approximation of $\sigma_{\theta}$ i.e. $[\sigma_{\theta}]_2$. Using $[\sigma_{\theta}]_2$ for $\sigma_{\theta}$ in Eq. (18), second approximation of $\sigma_i$, i.e. $[\sigma_i]_2$ is found and then the second approximation of $x$ i.e. $[x]_2$ is obtained. The iteration is continued till the process converges and gives the values of stresses at different points of the radius grid.

For rapid convergence 75 percent of the value of $\sigma_{\theta}$ obtained in the current iteration has been mixed with 25 percent of the value of $\sigma_{\theta}$ obtained in the last iteration for use in the next iteration i.e $\sigma_{\theta_{new}} = .25\sigma_{\theta_{previous}} + .75\sigma_{\theta_{current}}$. The strain rates are then calculated now from the Equations (8), (9) and (10).
4. Numerical computations and discussions

For the sake of computation of a rotating disc made of SiC\(_p\) reinforced 6061Al matrix composite, one has chosen particle size \( P = 1.7 \mu m \), particle content \( V = 30\% \) and temperature \( T = 623 K \). For a composite disc by reinforcing SiC\(_p\) in 6061Al matrix, the inner radii \( a \) and the outer radii \( b \) of all the discs are taken as 31.75mm and 152.4mm respectively. The stress exponent and Density of disc material have been taken as \( n = 8 \) and \( \rho = 2862.1 kg/m^3 \) respectively. For an anisotropic material, \( F/G = 1 \) and \( \frac{G}{H} = \frac{2}{\alpha^2} - 1 \) has been carried out. The isotropic material corresponds to \( \alpha = 1 \) and the effect of a deviation of \( \alpha \) from one on the resulting stress and creep will be investigated in the present study. For the analysis, the tensile residual stress (\( \Delta \sigma_y \)) is taken as 32 MPa, as observed by Rattan[10]. Linearly varying thickness i.e. \( h_a = 1.44 \), \( h_b = 0.75 \) has been taken, where \( h \) be the thickness of disc (mm) and the thickness \( h \) is assumed to be of the form,

\[
h = h_a + 2c (b - r) \quad \text{and} \quad h_a = h_b + 2c (b - a),
\]

where \( c \) is the slope of a disc and the values of \( c \) appearing in above equation are calculated as 0.002859.

\[
A = (r-a) [h_b + c (2b - r - a)] \quad \text{and} \quad A_b = (b-a) [h_b + c (b - a)]
\]

\[
I = \frac{h_b}{3} (r^3 - a^3) + \frac{2cb}{3} (r^3 - a^3) - \frac{c}{2} (r^4 - a^4) \quad \text{and} \quad I_o = \frac{h_b}{3} (b^3 - a^3) + \frac{2cb}{3} (b^3 - a^3) - \frac{c}{2} (b^4 - a^4)
\]

In presence of residual stress, the variation of stresses and strain rates in an anisotropic disc made of Al-SiC(particle) composite having linearly varying thickness and characterized by the parameter \( \alpha \) deviating from unity, has been investigated using anisotropic Hoffman yield criterion and results obtained are compared with those using Hill’s criterion ignoring difference in yield stresses i.e. (\( \Delta \sigma_y = 0 \)). Since it is assumed that \( F/H = 1 \) even for anisotropic case in terms of single parameter \( \alpha \). The value of \( \alpha \) less than unity or greater than unity indicating strengthening or weakening respectively in the tangential direction of the disc.

In figure 1, the variation of tangential stress along the radius in the anisotropic disc (\( \alpha = 0.7 \) and \( \alpha = 1.3 \)) have been investigated and the results are compared with an isotropic disc (\( \alpha = 1.0 \)). If \( \alpha \) is less than unity, the tangential stress reduce in the middle region of the disc in comparison to that in isotropic case, but enhance near the inner and outer part of disc. If \( \alpha \) is greater than unity, the tangential stress enhance in the middle region of the disc in comparison to that in isotropic case, but reduce near the inner and outer part of disc.
Figure 1: Variation of tangential Stress along the radial distance of the isotropic/anisotropic discs rotating with an angular velocity 15000 rpm at 623K.

Figure 2: Variation of radial Stress along the radial distance of the isotropic/anisotropic discs rotating over an angular velocity 15000 rpm at 623K.
Figure 3 Variation of tangential strain rate along the radial distance of the isotropic/anisotropic discs rotating with an angular velocity 15000 rpm at 623K.

In figure 2, the variation of radial stress along the radius in the anisotropic disc ($\alpha=0.7$ and $\alpha=1.3$) have been investigated and the results are compared with an isotropic disc ($\alpha=1.0$). If $\alpha$ is less than unity, the radial stress enhance in the region near inner radius of the disc in comparison to that in isotropic case, but reduce near the outer radius of disc. If $\alpha$ is greater than unity, the radial stress reduce in the region near inner radius of the disc in comparison to that in isotropic case, but enhance near the outer radius of disc. However, the magnitude of this change in tangential and radial stress is only a few percent. Therefore, it is concluded that stress distribution may not change significantly due to anisotropy introduced by processing or inhomogeneous distribution of reinforcement.

In figure 3, the variation of tangential strain rates along the radius in the anisotropic disc ($\alpha=0.7$ and $\alpha=1.3$) have been investigated and the results are compared with an isotropic disc ($\alpha=1.0$). The tangential strain rate is maximum for $\alpha$ greater than unity and minimum for $\alpha$ less than unity in comparison to that in isotropic case ($\alpha=1.0$). However, the trend of variation of tensile strain rate in tangential direction remains the same although the magnitude reduces by seven order of magnitude, if $\alpha$ is reduced from 1.3 to 0.7 i.e., the strength in tangential direction is enhanced.
Figure 4 Variation of radial strain rate along the radial distance of the isotropic/anisotropic discs rotating over an angular velocity 15000 rpm at 623K.

In figure 4, the variation of tangential strain rates along the radius in the anisotropic disc (\(\alpha = 0.7\) and \(\alpha = 1.3\)) have been investigated and the results are compared with an isotropic disc (\(\alpha = 1.0\)). It is observed that the radial strain rate which always remained compressive for the case where \(\alpha\) is greater than unity and \(\alpha\) is equal to unity, becomes tensile in the middle region of the disc when \(\alpha\) is less than unity. This is the consequence of the change in sign of \(\phi(r)\) as given in eq. (11) and this is responsible of the resulting tensile radial strain rate as given by eq. (8). This change in strain distribution due to the anisotropy in the disc with varying thickness may cause unwanted deformation in shape.

5. CONCLUSION

The above results and discussion concludes that

1. The anisotropy of material helps in restraining the creep in the rotating disc with thermal residual stress.
2. In case of anisotropy lowering the strength in tangential direction (α < 1), the tangential stress reduces in middle region of disc with linearly varying thickness, but enhances near the inner and outer region of the disc in comparison to that in isotropic disc.

3. If α is less than unity, In presence of residual stress the radial stress increases near the inner region, while it decreases near the outer region of the disc in comparison to those for isotropic composite.

4. The trend of variation of tensile strain rate in tangential direction remains the same by reducing α from 1.3 to 0.7, but the magnitude is reduced by about seven orders of magnitude.

5. The radial strain rate which always remained compressive for α greater than unity (α = 1.3) and for the isotropic case (α = 1), becomes tensile in the middle region of the disc for α is less than unity (α = 0.7).

6. The anisotropy introduces significant change in the strain rates, but its effect on the stresses distribution may be relatively small.

7. In presence of thermal residual stress, the material with anisotropy parameter (α < 1) seems justified for the safe design of the disc with linearly varying thickness as in this case stress and strain rate are lowest among all the cases for (α = 1.3) and (α = 0.7).

REFERENCES


