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THE DIAGONALIZATION MATRIX OF THE $\overset{n}{\otimes}(\equiv^* T_{p,q})$

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ABSTRACT

The aim of this paper is to determine the diagonalization of the $\overset{n}{\otimes}(\equiv^* T_{p,q})$*, where* $\overset{n}{\otimes}(\equiv^* T_{p,q})$ *is the tensor product of the matrix of the rational valued character table of the group* $T_{p,q}$ *by itself n- times, where p, q are prime number, p>q and* $q \mid p-1$.

Keywords: Tensor Product, The rational character table, the group $T_{p,q}$

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1. INTRODUCTION

The tensor product of two matrices and the rational character table of the group $T_{p,q}$ has been given in respectively [1], [5].

In this work, we found two matrices P, Q and we give some concepts that we shall use to determine the diagonal matrix of the tenser product of the matrix rational character table of group $T_{p,q}$ of n- times of itself where p, q are prime number, p>q and q | p−1.

Preliminaries

Some definition and basic concepts have been given in this section

Definition (2-1), [1]: Let $A \in M_n(K)$, $B \in M_m(K)$, we define a matrix $A \otimes B \in M_{nm}(K)$ by :

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots a_{1n}B \\ a_{21}B & a_{22}B & \dots a_{2n}B \\ \vdots & \vdots & \dots \vdots \\ a_{n1}B & a_{n1}B & \cdots a_{nn}B \end{bmatrix}_{nm \times mn}$$

Where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \dots \vdots \\ a_{n1} & a_{n1} & \cdots a_{nn} \end{bmatrix}_{n \times n} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \dots b_{1n} \\ b_{21} & b_{22} & \dots b_{2n} \\ \vdots & \vdots & \dots \vdots \\ b_{n1} & b_{n1} & \cdots b_{nn} \end{bmatrix}_{m \times m}$$

Thus,

$$A \otimes B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots \alpha_{2n} \\ \vdots & \vdots & \cdots \vdots \\ \alpha_{n1} & \alpha_{n1} & \cdots \alpha_{nn} \end{bmatrix}$$

Where

$$\alpha_{11} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots a_{11}b_{1m} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots a_{11}b_{2m} \\ \vdots & \vdots & \cdots \vdots \\ a_{11}b_{m1} & a_{11}b_{m2} & \cdots a_{11}b_{mm} \end{bmatrix}_{m\times m}$$

$$\alpha_{1k} = \begin{bmatrix} a_{1n}b_{11} & a_{1n}b_{12} & \cdots a_{1n}b_{1m} \\ a_{1n}b_{21} & a_{1n}b_{22} & \cdots a_{1n}b_{2m} \\ \vdots & \vdots & \cdots \vdots \\ a_{1n}b_{m1} & a_{1n}b_{m2} & \cdots a_{1n}b_{mm} \end{bmatrix}_{m\times m}$$

$$\alpha_{kk} = \begin{bmatrix} a_{nn}b_{11} & a_{nn}b_{12} & \cdots a_{nn}b_{1m} \\ a_{nn}b_{21} & a_{nn}b_{22} & \cdots a_{nn}b_{2m} \\ \vdots & \vdots & \cdots \vdots \\ a_{nn}b_{m1} & a_{nn}b_{m2} & \cdots a_{nn}b_{mm} \end{bmatrix}_{m\times m} \quad \text{and } k=nm$$

Example (2-2): Consider $A=\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$, then

$$A \otimes B = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & -2 \end{bmatrix}_{6\times 6}$$

Proposition (2-3), [1]:

Let A ,A` be two different matrices in $M_n(K)$ and B , B` be two different matrices in $M_m(K)$, then

1- $(A + A^{`}) \otimes B = (A\otimes B) + (A^{`}\otimes B)$.
2- $(A\otimes B) . (A^{`}\otimes B^{`}) = AA^{`}\otimes BB^{`}$.
3- $\det(A\otimes B) = (\det(A))^{n} . (\det(B))^{m}$.

Definition (2-4), [2]:

Let T be a matrix representation of finite group G over a field F , then **the character χ** of T is a mapping $\chi : G \longrightarrow F$ define by $\chi(g)=Tr(T(g))$ refers to the trace of the matrix T(g) .

Clearly $\chi(1) = n$, which is called the **degree of χ** . Also, characters of degree 1 are called **linear characters**.

Example (2-5):

In symmetric group $S_3=< x,y : x^2=y^3=1, xy= y^2x>$, define the representation $T : S_3 \longrightarrow GL(2 , \mathbb{C})$ such that : $T(x)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $T(y)=\begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}$, where $w = e^{2\pi i/3}$, then the character χ of T is

$\chi(T(x)) = 0+0=0$, $\chi(T(y)) = w + w^2 = -1$

Definition (2-6), [2]:

The character afforded by irreducible representation is called **irreducible character**; otherwise it is called **compound character**.

Example (2-7): Linear characters are irreducible character.

Definition (2-7), [3]:

A class function on a group G is a function $f : G \longrightarrow \mathbb{C}$ which is constant on conjugate classes , that is $f(x^{-1}y\,x) = f(y)$, $\forall$ x ,y $\in$ G , if all values of f are in Z , then it is called **Z-valued class function** .

Proposition (2-8), [3]: characters are class function.

Proof:

Let T be matrix representation and χ character of T, then

$\chi(x^{-1}y\,x) = Tr(T(x^{-1}y\,x)) = Tr(T(x^{-1})\,T(y)\,T(x))$

$= Tr(T(x^{-1})\,T(x)\,T(y))$

$= Tr(T(y)) = \chi(y)$

Proposition (2-9), [5]:

Let p and q be two prime numbers such that p>q and q|p-1, then the rational character table of the group $T_{p,q}$ is

($\equiv^* T_{p,q}$)=

	K_1	K_2	K_3
γ_1	1	1	1
γ_2	q−1	q−1	−1
γ_3	p−1	−1	0

Definition (2-10), [3]:

A **rational valued** character $\theta$of G is a character whose values are in Z,That is $\theta(x) \in Z, \forall x \in G$.

Theorem (2-11), [6]:

Let M be anm $\times$ n matrix with entries in a principal domain R, then thereexist matricesP,Q, D such that:

1- P and Q are invertible
2- QMP^{-1}=D
3- D is diagonal matrix
4- If we denoted D_{ii} by d_i , then there exists a natural number r , $0 \leq r \leq \min(m, n)$ such that j>r implies $d_j = 0$ and j$\leq$ r implies $d_j \neq 0$ and $1 \leq j \leq r$ implies d_j divides d_{j+1} .

Definition (2-12), [6]:

Let M be a matrix with entries in a principal domain R , be equivalent to a matrixD= diag{ d_1 , d_2 , ..., d_r ,0 ,0 , ..., 0} such that d_j / d_{j+1} for $1 \le j \le r$, we call **D** the **invariant factor** matrix of M and $\mathbf{d_1}$ **,** $\mathbf{d_2}$ **, ...,** $\mathbf{d_r}$ the invariant factor of **M** .

Theorem (2-12), [6]: Let Mbe a matrix with entries in a principal domain R, thenthe invariant factor areunique.

Theorem (2-13), [4]:

Let A, B are two matrices nonsingular matrices of degree n, m respectively over principal domain R, and let

$P_1 A\, Q_1 = D(A)$ =Diag { $d_1(A), d_2(A), \cdots, d_n(A)$},

$P_2 B\, Q_2 = D(B)$ =Diag { $d_1(B), d_2(B), \cdots, d_n(B)$}, be the invariant factor matrices of A and B,

Then, $(P_1 \otimes P_2).(A \otimes B).(Q_1 \otimes Q_2) = D(A) \otimes D(B)$

And, from this the invariant factor matrices of A⊗ B can be written down

Let H and L be P_1 and P_2−groups respectively, where P_1 and P_2 are distinct primes, we know that:

$\equiv(H \times L) = \equiv(H) \otimes \equiv(L)$,since $\gcd(P_1, P_2)=1$, we have

$\equiv^{*}(H \times L) = \equiv^{*}(H) \otimes \equiv^{*}(L)$.

The Diagonal Matrix of The $\overset{n}{\otimes}$($\equiv^{*}$ $\mathbf{T_{p,q}}$):

In this section, we found two matrices P and Qto determine the diagonal matrix of the $\overset{n}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$), where $\overset{n}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$) denote to the tenser product of the matrix rational character table of group$T_{p,q}$ by itself n- times. Where p, q are prime number, $p>q$ and $q \mid p-1$.

We apply theorem (2-13) to determine the diagonal of $\overset{n}{\otimes}$($\equiv^{*}$ $T_{p,q}$) .

Let p= $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ and Q= $\begin{bmatrix} 1 & p & 1 \\ -1 & 0 & -1 \\ -p & -p & 0 \end{bmatrix}$

be two matrices which is the invariant factor matrix for $\equiv^{*}$ $T_{p,q}$ where

$$\equiv^{*} T_{p,q} = \begin{bmatrix} 1 & 1 & 1 \\ q-1 & q-1 & -1 \\ p-1 & -1 & 0 \end{bmatrix}$$

Hence, by theorem (2-13) we get

$$P.(\equiv^{*} T_{p,q}).Q = \begin{bmatrix} -p & 0 & 0 \\ 0 & -qp & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Hence, by Theorem (2-13) $\Longrightarrow$ D(($\equiv^{*}$ $T_{p,q}$))= diag {$-p$, $-qp$, $-p$ }

Now, we consider explicitly the case n=2, then

$$P\otimes P=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}_{9\times 9},$$

$$Q\otimes Q=\begin{bmatrix} 1 & P & 1 & P & P^2 & P & 1 & P & 1 \\ -1 & 0 & -1 & -P & 0 & -P & -1 & 0 & -1 \\ -P & -P & 0 & -P^2 & -P^2 & 0 & -P & -P & 0 \\ -1 & -P & -1 & 0 & 0 & 0 & -1 & -P & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ P & P & 0 & 0 & 0 & 0 & P & P & 0 \\ -P & -P^2 & -P & -P & -P^2 & -P & 0 & 0 & 0 \\ P & 0 & P & P & 0 & P & 0 & 0 & 0 \\ P^2 & P^2 & 0 & P^2 & P^2 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 9}$$

And, $(\equiv^* T_{p,q})\otimes(\equiv^* T_{p,q})=\overset{2}{\otimes}(\equiv^* T_{p,q})=$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ q-1 & -1 & q-1 & q-1 & -1 & q-1 & q-1 & -1 \\ p-1 & 0 & p-1 & -1 & 0 & p-1 & -1 & 0 \\ q-1 & q-1 & q-1 & q-1 & q-1 & -1 & -1 & -1 \\ (q-1)^2 & 1-q & (q-1)^2 & (q-1)^2 & 1-q & 1-q & 1-q & -1 \\ (q-1)(p-1) & 0 & (q-1)(p-1) & 1-q & 0 & 1-p & 1 & 1 \\ p-1 & p-1 & -1 & -1 & -1 & 0 & 0 & 0 \\ (q-1)(p-1) & 1-p & 1-q & 1-q & 1 & 0 & 0 & 0 \\ (p-1)^2 & 0 & 1-p & 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 9}$$

So, we obtain

$$(P\otimes P).\,(\overset{2}{\otimes}(\equiv^* T_{p,q}))\,.\,(Q\otimes Q)=\begin{bmatrix} p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & qp^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & qp^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2p^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & qp^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & qp^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 \end{bmatrix}_{9\times 9}$$

Hence, by Theorem (2-13) $\Longrightarrow$ $D(\overset{2}{\otimes}(\equiv^* T_{p,q}))=\text{diag}\{p^2, qp^2, p^2, qp^2, q^2p^2, qp^2, p^2, qp^2, p^2\}$

We, consider explicitly the case n=3, then we obtain

$(P\otimes P\otimes P).\,(\overset{3}{\otimes}(\equiv^* T_{p,q}))\,.\,(Q\otimes Q\otimes Q)=D(\overset{3}{\otimes}(\equiv^* T_{p,q}))=$

$$\begin{bmatrix}
-p^3 \\
& -qp^3 \\
& & -p^3 \\
& & & -qp^3 \\
& & & & -q^2p^3 \\
& & & & & -qp^3 \\
& & & & & & -p^3 \\
& & & & & & & -qp^3 \\
& & & & & & & & -p^3 \\
& & & & & & & & & -qp^3 \\
& & & & & & & & & & -q^2p^3 \\
& & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & -q^2p^3 \\
& & & & & & & & & & & & & -q^3p^3 \\
& & & & & & & & & & & & & & -q^2p^3 \\
& & & & & & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & & & & & -q^2p^3 \\
& & & & & & & & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & & & & & & & -p^3 \\
& & & & & & & & & & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & & & & & & & & & -p^3 \\
& & & & & & & & & & & & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & & & & & & & & & & & -q^2p^3 \\
& & & & & & & & & & & & & & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & & & & & & & & & & & & & -p^3 \\
& & & & & & & & & & & & & & & & & & & & & & & & & -qp^3 \\
& & & & & & & & & & & & & & & & & & & & & & & & & & -p^3
\end{bmatrix}_{27\times 27}$$

Hence, by Theorem (2-13) we get:

D($\overset{3}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$))= diag {$-p^3$,$-qp^3$,$-p^3$,$-qp^3$,$-q^2p^3$,$-qp^3$,$-p^3$,$-qp^3$,$-p^3$,$-q\,p^3$, $-q^2p^3$, $-qp^3$, $-q^2p^3$, $-q^3p^3 - q^2p^3$,$-qp^3$, $-q^2p^3$, $-qp^3$, $-p^3$,$-qp^3$,$-p^3$, qp^3,$-q^2p^3$,$-qp^3$,$-p^3$,$-qp^3$,$-p^3$} .

We, consider explicitly the case n=4, then we obtain

(P⊗P⊗P⊗P). ($\overset{4}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$)). (Q⊗Q⊗Q⊗Q)=D($\overset{4}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$))=

$$\begin{bmatrix}
(-p \\
& , \\
& & -qp \\
& & & , \\
& & & & -p\,) \\
& & & & & \otimes \\
& & & & & & (\overset{\mathbf{3}}{\otimes} \\
& & & & & & & (\,-p \\
& & & & & & & & , \\
& & & & & & & & & -qp \\
& & & & & & & & & & , \\
& & & & & & & & & & & -p\,)
\end{bmatrix}_{81\times 81}$$

Therefore, (P⊗P⊗P⊗P). ($\overset{4}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$)). (Q⊗Q⊗Q⊗Q)=D($\overset{4}{\otimes}$ ($\equiv^{*}$ $T_{p,q}$))= diag { p^4, qp^4 , p^4,qp^4,q^2p^4, qp^4,p^4,qp^4 ,p^4 ,$q\,p^4$, q^2p^4, qp^4, q^2p^4,$q^3p^4q^2p^4$, qp^4,q^2p^4, qp^4, p^4, qp^4 , p^4,qp^4,q^2p^4, qp^4,p^4,qp^4 ,p^4 ,$q\,p^4$, q^2p^4, qp^4, q^2p^4,q^3p^4 , q^2p^4, qp^4,q^2p^4, qp^4, q^2p^4, q^3p^4,q^2p^4 ,q^3p^4, q^4p^4,q^3p^4 , q^2p^4, q^3p^4,q^2p^4 , p^4,q^2p^4 , p^4,,q^2p^4,q^3p^4 , q^2p^4, qp^4,q^2p^4, qp^4, p^4, qp^4 , p^4,qp^4,q^2p^4, qp^4,p^4,qp^4,p^4 ,$q\,p^4$, q^2p^4, qp^4, q^2p^4,$q^3p^4q^2p^4$, qp^4,q^2p^4, qp^4, p^4, qp^4 , p^4,qp^4,q^2p^4, qp^4,p^4,qp^4 ,p^4}.

The general case for p, q are prime number, p>q and q | p−1 given by the following proposition.

Proposition: If $p>q$ and $q \mid p-1$, then

$$(\overset{n}{\otimes}P).(\overset{n}{\otimes}(\equiv^* T_{p,q})).(\overset{n}{\otimes}Q)=\text{diag}\left\{(-p,-qp,-p)\otimes\left(\overset{\mathbf{n-1}}{\otimes}(-p,-qp,-p)\right)\right\}=D(\overset{n}{\otimes}(\equiv^* T_{p,q}))$$

Proof: By an inductive argument, the statement is certainly true for k=1

Assuming it holds for an arbitrary k, then $(\overset{k}{\otimes}P).(\overset{k}{\otimes}(\equiv^* T_{p,q})).(\overset{k}{\otimes}Q)=D(\overset{k}{\otimes}(\equiv^* T_{p,q}))$

By theorem (2-13), we obtain

$$(\overset{k+1}{\otimes}(\equiv^* T_{p,q}))=(\overset{k}{\otimes}(\equiv^* T_{p,q}))\otimes((\equiv^* T_{p,q}))$$

Hence, $D(\overset{k+1}{\otimes}(\equiv^* T_{p,q}))=D(\overset{k}{\otimes}(\equiv^* T_{p,q}))\otimes D((\equiv^* T_{p,q}))$. □

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