INTEGRAL MEANS AND HOLDER INEQUALITIES OF ANALYTIC FUNCTIONS ASSOCIATED WITH FRACTIONAL Q-CALCULUS OPERATORS

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ABSTRACT

In this paper, we introduce a new class of analytic and univalent functions with negative coefficients based on fractional q-calculus operators and obtained the coefficient estimates, Holder inequality and results on integral means.

AMS Subject Classification: [2010] Primary 30C45; Secondary 30C50.

Key words: Analytic Functions, Starlike Functions, Hypergeometric Functions, Fractional q-Calculus Operators.


1. INTRODUCTION AND BASIC DEFINITIONS

The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference, q-integral equations and in q-transform analysis (see [3], [1, 2, 5]).

The q-shifted factorial is defined for \( \alpha, q \in \mathbb{C} \) as a product of \( n \) factors by

\[
(\alpha; q)_n = \begin{cases} 
1 & ; \ n = 0 \\
(1 - \alpha)(1 - \alpha q) \cdots (1 - \alpha q^{n-1}) & ; \ n \in \mathbb{N},
\end{cases}
\]

and in terms of the basic analogue of the gamma function

\[
(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n > 0),
\]

where the q-gamma function is defined by ([3], p. 16, eqn. (1.10.1))
\[ \Gamma_q(x) = \frac{(q; q)_\infty (1 - q)^{1-x}}{(q^x; q)_\infty} \quad (0 < q < 1). \]  

(3)

If \(|q| < 1\), the equation (1) remains meaningful for \(n = \infty\) as a convergent infinite product:

\[ (\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j). \]

We recall here the following \(q\)-analogue definitions Jackson’s \(q\)-derivative and \(q\)-integral of a function \(f\) defined on a subset of \(C\) are, respectively, due to Gasper and Rahman [3]

\[ D_{q,z}f(z) = \frac{f(z) - f(qz)}{z(1 - q)} \quad (z \neq 0, q \neq 0) \]

(4)

and

\[ \int_{q^0}^z f(t) d(t; q) = z(1 - q) \sum_{k=0}^{\infty} q^k f(q^k) . \]

(5)

In view of the relation

\[ \lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1 - q)^n} = (\alpha)_n , \]

(6)

we observe that the \(q\)-shifted factorial (1) reduces to the familiar Pochhammer symbol \((\alpha)_n\), where \((\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)\).

We now recall the fractional \(q\)-calculus operators defined and discussed by Purohit and Raina [9]

**Definition 1 (Fractional \(q\)-Integral Operator).** The fractional \(q\)-integral operator \(I_{q,z}^\alpha f(z)\) of a function \(f(z)\) of order \(\alpha\) is defined by

\[ I_{q,z}^\alpha f(z) \equiv D_{q,z}^{-\alpha} f(z) = \frac{1}{\Gamma_q(\alpha)} \int_{q^0}^z (z - tq)_{\alpha-1} f(t) d(t; q) \quad (\alpha > 0), \]

(7)

where \(f(z)\) is analytic in a simply-connected region of the \(z\)-plane containing the origin.

**Definition 2 (Fractional \(q\)-Derivative Operator).** The fractional \(q\)-derivative operator \(D_{q,z}^\alpha f(z)\) of a function \(f(z)\) of order \(\alpha\) is defined by

\[ D_{q,z}^\alpha f(z) = D_{q,z} I_{q,z}^{1-\alpha} f(z) \]

(8)

\[ = \frac{1}{\Gamma_q(1 - \alpha)} D_{q,z} \int_{q^0}^z (z - tq)^{1-\alpha} f(t) d(t; q) . (0 \leq \alpha < 1), \]

where \(f(z)\) is suitably constrained and the multiplicity of \((z - tq)^{1-\alpha}\) is removed as in Definition 1 above.
Definition 3 (Extended Fractional $q$-Derivative Operator). Under the hypotheses of Definition 2, the fractional $q$-derivative for a function $f(z)$ of order $\alpha$ is defined by

$$D^\alpha_{q,z} f(z) = D^m_{q,z} f(z) \frac{1}{m!} \sum_{k=0}^{\infty} \frac{\Gamma_q(\alpha)}{\Gamma_q(m+\alpha)} (m-1) z^{m-1} f(z),$$

where $N$ denotes the set of natural numbers.

We give the following image formulas for the function $z^\lambda$ under the fractional $q$-integral and $q$-differential operators defined by (7) and (8).

**Proposition 1.** Let $\alpha > 0$ and $\lambda > -1$, then

$$I^\alpha_{q,z} z^\lambda = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\alpha+1)} z^{\alpha+\lambda}. \quad (10)$$

**Proposition 2.** Let $\alpha \geq 0$ and $\lambda > -1$, then

$$D^\alpha_{q,z} z^\lambda = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)} z^{\lambda-\alpha}. \quad (11)$$

Let $A_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N}), \quad (12)$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$. Also, let $A^-_n$ denote the subclass of $A_n$ consisting of analytic and univalent functions expressed in the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0, n \in \mathbb{N}). \quad (13)$$

We note that $A^- = T$. Recently Making use of this $q$-calculus operator, Purohit and Raina[9] defined a fractional $q$-differ-integral operator $\Omega^\alpha_{q,z}$ and it is expressed as

$$\Omega^\alpha_{q,z} f(z) = 1 + \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha) \Gamma_q(k+1)}{\Gamma_q(2) \Gamma_q(k+1-\alpha)} a_k z^{k-1}$$

$$= \frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)} z^{\alpha-1} D^\alpha_{q,z} f(z) \quad (\lambda \in N; 0 < q < 1; z \in U). \quad (14)$$
Motivated with the work of Murugusundaramoorthy et al. [7] in this paper we introduce a new class of analytic functions involving the operator $\Omega^a_{q,z}$ is given by

$$J^a_{q,\delta, f}(z) = \left \{ f \in A_q, \left| \frac{(1-q^{1-\alpha})}{1-q} \frac{z\Omega^a_{q,z}f(z)}{\Omega^a_{q,z}f(z)} - 1 - \beta \left( \frac{1-q^{1-\alpha}}{1-q} \frac{z\Omega^a_{q,z}f(z)}{\Omega^a_{q,z}f(z)} - 2\delta + 1 \right) \right| < \beta \right \}$$

(15)

$$(-\infty < \alpha < 2, 0 \leq \delta < 1, 0 \leq \beta < 1, 0 < q < 1, z \in U)$$

Considering the ideas from Murugusundaramoorthy et al. [6] and Rosy et al. [11] in the following sections we discussed the coefficient inequalities, integral means results and Holder inequalities for the functions belonging to the class $J^a_{q,\delta}$.

2. COEFFICIENT INEQUALITIES

**Theorem 1.** A function $f$ of the form (13) belongs to the class $J^a_{q,\delta}$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k-\alpha)\Gamma_q(2)} (1+\beta)a_k \leq 2\beta(1-\delta).$$

The result is sharp for

$$f(z) = z - \frac{2\beta(1-\delta)\Gamma_q(2)\Gamma_q(n-\alpha+1)}{(1+\beta)\Gamma_q(n+2)\Gamma_q(2-\alpha)} z^{n+1} \quad (n \in \mathbb{N}).$$

**Proof.** Assume that the inequality (16) holds true and let $|z| = 1$, then on using (13) and (15), we find that

$$\left| \frac{1-q^{1-\alpha}}{1-q} \frac{z\Omega^a_{q,z}f(z)}{\Omega^a_{q,z}f(z)} - 1 - \beta \left( \frac{1-q^{1-\alpha}}{1-q} \frac{z\Omega^a_{q,z}f(z)}{\Omega^a_{q,z}f(z)} - 2\delta + 1 \right) \right|$$

$$= -\sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k-\alpha)} a_k z^{k-1}$$

$$- \beta(1-\delta) - \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k-\alpha)} a_k z^{k-1}$$

$$\leq \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k-\alpha)} (1+\beta)a_k - 2\beta(1-\delta) \leq 0,$$

by our hypothesis. This implies that $f(z) \in J^a_{q,\delta}$. 

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Integral Means and Holder Inequalities of Analytic Functions Associated with Fractional Q-Calculus Operators

To prove the converse, assume that \( f(z) \) is defined by (13) and is in the class \( J_{q,\delta}^\alpha \), then it follows that

\[
\left| \frac{\left(1 - q^{1-\alpha}\right)}{1-q} z \Omega_{q,z}^{\alpha+1} f(z) - 1 \right| = \left| \sum_{k=n+1}^\infty \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k-\alpha)} a_k z^{k-1} \right| \leq 2\delta + 1
\]

\[
\times\left| 2(1-\delta) - \sum_{k=n+1}^\infty \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k-\alpha)} a_k z^{k-1} \right|^{1}\beta \leq 2\delta + 1
\]

Since \( \Re(z) \leq |z| \) for any \( z \), therefore, choosing values of \( z \) on the real axis so that \( \Omega_{q,z}^{\alpha} f(z) \) is real, and letting \( z \rightarrow 1^- \) through real values, we obtain from (18) we get the desired result (16).

### 3. HOLDER’S INEQUALITY

Followed by Nishiwaki et al.[10] and Murugusundaramoorthy et al.[8] in this section we study some results of Holder type inequalities for \( f \in J_{q,\delta}^\alpha \). Now we recall the generalisation of the convolution as given below

\[
H_m(z) = z - \sum_{n=2}^\infty \left( \prod_{j=1}^m a_{n,j}^p \right) z^n, \quad (p_j > 0, j = 1, 2, \ldots, m).
\]

(19)

Further for functions \( f_j \in J_{q,\delta}^\alpha f(z), \quad (j = 1, 2, \ldots, m) \)

\[(-\infty < \alpha < 2, 0 \leq \delta < 1, 0 \leq \beta < 1, 0 < q < 1, z \in \mathbb{U})\]

Given by the familiar Holder inequality assumes the following form

\[
\sum_{n=2}^\infty \left( \prod_{j=1}^m a_{n,j}^p \right) \leq \left( \prod_{n=2}^\infty \left( \sum_{j=1}^m a_{n,j}^p \right) \right)^{1/p},
\]

(20)

\[(p_j > 1, j = 1, 2, \ldots, m, \sum_{j=1}^m \frac{1}{p_j} \geq 1).\]

**Theorem 2** If \( f_j \in J_{q,\delta}^\alpha f(z), \quad (j = 1, 2, \ldots, m) \) then \( H_m(z) \in J_{q,\delta}^\alpha f(z) \) with

\[
\xi \leq 1 - \frac{2\beta}{\prod_{j=1}^m (1-\xi_j)^{p_j} (1-(1-\beta)2\beta)} \]

\[
\prod_{j=1}^m \left[ (2\beta^2 - \xi_j) + (1) \right]^{p_j} - [2\beta]^{m} \prod_{j=1}^m (1-\xi_j)^{p_j}
\]

where

\[
s = \sum_{j=1}^m p_j > 1; \quad p_j \geq \frac{1}{q_j} \quad (j = 1, 2, \ldots, m), q_j > 1 (j = 1, 2, \ldots, m); \quad \sum_{j=1}^m q_j \geq 1.
\]
Proof. Let $f_j \in J_q^j(\alpha, \beta, \gamma, A, B), (j = 1, 2, \ldots, m)$ then we have

$$\sum_{n=2}^{\infty} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \leq 1,$$

which in turn implies that

$$\left( \sum_{n=2}^{\infty} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \right)^{\frac{1}{q_j}} \leq 1.$$

Applying the inequality (20) we arrive at the following inequality

$$\sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \right)^{\frac{1}{q_j}} \leq 1.$$

Thus we determine the largest $\xi$ such that

$$\left( \sum_{n=2}^{\infty} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \right)^{\frac{1}{q_j}} \leq 1.$$

That is

$$\left( \sum_{n=2}^{\infty} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \right)^{\frac{1}{q_j}} \leq 1.$$

We see that

$$\prod_{j=1}^{m} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \leq 1,$$

This last inequality (22) implies that

$$2 \beta(1 - \delta) \Gamma_q(2) \prod_{j=1}^{m} \frac{\Gamma_q(n + 1) \Gamma_q(2 - \xi_j)(1 + \beta)}{2 \beta(1 - \delta) \Gamma_q(k - \xi_j) \Gamma_q(2)} a_{n,j} \leq 1.$$
Integral Means and Holder Inequalities of Analytic Functions Associated with Fractional Q-Calculus Operators

\[
\leq \left\{ -\beta \Gamma_q(k+1)\Gamma_q(2-\xi_j)\prod_{j=1}^{m}(2\beta(1-\delta))^{\gamma_j-1}\Gamma_q(k-\xi_j)^{\gamma_j} \right\} \\
+ \left\{ \Gamma_q(k+1)\Gamma_q(2-\xi_j)\prod_{j=1}^{m}(2\beta(1-\delta))^{\gamma_j-1}\Gamma_q(k-\xi_j)^{\gamma_j} \right\},
\]

Where

\[
Y_j = \prod_{j=1}^{m}(2\beta(1-\delta))^{\gamma_j}(\Gamma_q(k-\xi_j)^{\gamma_j}).
\]

Hence we have

\[
\left[ Y_j - \sum_{j=1}^{m}[\Gamma_q(k+1)\Gamma_q(2-\xi_j)(1+\beta)]^{\gamma_j} \right] (1-\xi_j) \leq \Gamma_q(k+1)\Gamma_q(2-\xi_j)\left[ Y_j - \beta \prod_{j=1}^{m}(2\beta\gamma(B-A))^{\gamma_j-1}(1-\xi_j)^{\gamma_j} \right].
\]

That is

\[
\xi \leq 1 - \frac{\sum_{j=1}^{m}[\Gamma_q(k+1)\Gamma_q(2-\xi_j)(1+\beta)]^{\gamma_j} - Y_j}{\sum_{j=1}^{m}[\Gamma_q(k+1)\Gamma_q(2-\xi_j)(1+\beta)]^{\gamma_j}} = \Phi(k).
\]

which is an increasing function in \(k\) hence for \(k = 2\) we have (21), which completes the proof.
Hence the proof.

4. INTEGRAL MEANS INEQUALITIES

In this section, we obtain integral means inequalities for the functions in the family \(J^{q,\delta}_{q,\eta}\).

**Lemma 1:** (Littlewood[4]) If the functions \(f\) and \(g\) are analytic in \(U\) with \(g < f\), then for \(\eta > 0\), and \(0 < r < 1\),

\[
\int_0^{2\pi} g(re^{i\theta})^\eta d\theta \leq \int_0^{2\pi} f(re^{i\theta})^\eta d\theta.
\]

In 1975, Silverman, found that the function \(f_z(z) = z - \frac{z^2}{2}\) is often extremal over the family \(T\) and applied this function to resolve his integral means inequality,

\[
\int_0^{2\pi} f(re^{i\theta})^\eta d\theta \leq \int_0^{2\pi} f_z(re^{i\theta})^\eta d\theta,
\]

for all \(f \in T\), \(\eta > 0\) and \(0 < r < 1\).

Applying Lemma 1, we prove the following result.
**Theorem 3:** Suppose $f \in J_{q,\delta}^{\alpha}, \eta > 0$, $(-\infty < \alpha < 2, 0 \leq \delta < 1, 0 \leq \beta < 1, 0 < q < 1, z \in \mathbb{U})$ and $f_{2}(z)$ is defined by

$$f_{2}(z) = z - \frac{2\beta(1-\delta)\Gamma_{q}(2)\Gamma_{q}(3-\alpha)}{(1+\beta)\Gamma_{q}(4)\Gamma_{q}(2-\alpha)}z^{2},$$

For $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_{0}^{2\pi} |f(z)|^{\eta} d\theta \leq \int_{0}^{2\pi} |f_{2}(z)|^{\eta} d\theta. \quad (24)$$

**Proof.** For given $f$ of the form (13), from (24) is equivalent to proving that

$$\int_{0}^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_{n}| |z|^{n-1} \right|^{\eta} d\theta \leq \int_{0}^{2\pi} \left| 1 - \frac{2\beta(1-\delta)\Gamma_{q}(2)\Gamma_{q}(3-\alpha)}{(1+\beta)\Gamma_{q}(4)\Gamma_{q}(2-\alpha)}z^{2} \right|^{\eta} d\theta. \quad (25)$$

By Lemma 1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_{n}| |z|^{n-1} - 1 - \frac{2\beta(1 - \delta)\Gamma_{q}(2)\Gamma_{q}(3 - \alpha)}{(1 + \beta)\Gamma_{q}(4)\Gamma_{q}(2 - \alpha)} z.$$ 

Setting

$$1 - \sum_{n=2}^{\infty} |a_{n}| |z|^{n-1} = 1 - \frac{2\beta(1-\delta)\Gamma_{q}(2)\Gamma_{q}(3-\alpha)}{(1+\beta)\Gamma_{q}(4)\Gamma_{q}(2-\alpha)} w(z) \quad (25)$$

and using (16), we obtain

$$|w(z)| \leq \sum_{n=2}^{\infty} \frac{(1 + \beta)\Gamma_{q}(4)\Gamma_{q}(2 - \alpha)}{2\beta(1 - \delta)\Gamma_{q}(2)\Gamma_{q}(3 - \alpha)} |a_{n}| |z|^{n-1} \leq |z| \sum_{n=2}^{\infty} \frac{(1 + \beta)\Gamma_{q}(4)\Gamma_{q}(2 - \alpha)}{2\beta(1 - \delta)\Gamma_{q}(2)\Gamma_{q}(3 - \alpha)} |a_{n}| \leq |z|.$$ 

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