NEW CHARACTERIZATION OF KERNEL SET IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT

In this paper we introduce a kernelled fuzzy point, boundary kernelled fuzzy point and derived kernelled fuzzy point of a subset $A$ of $X$, and using these notions to define kernel set of fuzzy topological spaces. Also we introduce fuzzy topological $kr$-space. The investigation enables us to present some new fuzzy separation axioms between $FT_0$ and $FT_1$-spaces.

Keywords: Fuzzy Topological Space, Kernelled Fuzzy Point, Boundary Kernelled Fuzzy Point, Derived Kernelled Fuzzy Point, Kernel Set, Weak Fuzzy Separation Axioms, $FR_i$-space, $i = 0, 1$.

1. INTRODUCTION

The concept of fuzzy set and fuzzy set operations were first introduced by L. A. Zadeh in 1965 [8]. After Zadeh’s introduction of fuzzy sets, Chang [3] defined and studied the notion of fuzzy topological space in 1968. In 1997, fuzzy generalized closed set ($Fg$-closed set) was introduced by G. Balasubramania and P. Sundaram [6]. In 1998, the notion of $Fgs$-closed set was defined and investigated by H. Maki et al [7]. In 2002, O. Bedre Ozbakir [11], defined the concept of fuzzy generalized strongly closed set. In 1984, fuzzy separation axioms have been introduced and investigated by A. S. Mashour and others [1].

In this paper we introduce a new characterization of kernel set through our definition kernelled fuzzy point, boundary kernelled fuzzy point and derived kernelled fuzzy point. By these notions, we obtain that the kernel of a set in fuzzy topological space $(X, T)$ is a union of the set itself with the set of all boundary kernelled fuzzy points. In addition, it is a union of the set itself with the set of all derived kernelled fuzzy points and we give some result of $FR_0$-space by using these notions. Also in this paper we introduce fuzzy topological $kr$-space iff kernel of a subset $A$ of $X$ is
an fuzzy open set. Via this kind of fuzzy topological space, we give a new characterization of fuzzy separation axioms lying between \( FT_0 \) and \( FT_1 \)-spaces.

2. PRELIMINARIES

Fuzzy sets theory, introduced by Lotfi. A. Zada in 1965 [8], is the extension of classical set theory by allowing the membership of elements to range from 0 to 1. Let \( X \) be the universe of a classical set of objects. Membership in a classical subset \( A \) of \( X \) is often viewed as a characteristic function \( \mu_A \) from \( X \) into \{0, 1\}, where

\[
\mu_A(x) = \begin{cases} 
1 & \text{for } x \in A \\
0 & \text{for } x \notin A
\end{cases}
\]

for any \( x \in X \).

\{0,1\} is called a valuation set (see [13]). If the valuation set is allowed to be the real interval \([0,1] \), \( A \) is called a fuzzy set in \( X \). \( \mu_A(x) \) (or simply \( A(x) \)) is the membership value (or degree of membership) of \( x \) in \( A \). Clearly, \( A \) is a subset of \( X \) that has no sharp boundary. A fuzzy set \( A \) in \( X \) can be represented by the set of pairs: \( A = \{ (x, A(x)), x \in X \} \).

Let \( A : X \to [0,1] \) be a fuzzy set. If \( A(x) = 1 \), for each \( x \in X \), we denote it by \( 1_A \) and if \( A(x) = 0 \), for each \( x \in X \), we denote it by \( 0_A \). That is, by \( 0_A \) and \( 1_A \), we mean the constant fuzzy sets taking the values 0 and 1 on \( X \), respectively [2]. Let \( I = [0,1] \). The set of all fuzzy sets in \( X \), denoted by \( I^X \) [10].

**Definition 2.1:** [4] Let \( A \) be a fuzzy set of a set \( X \). The support of \( A \) is the elements \( x \) whose membership value is greater than 0, i.e., \( \text{supp}(A) = \{ x \in X : A(x) > 0 \} \).

**Definition 2.2:** [5] Let \( A \) and \( B \) be any two fuzzy sets in \( X \). Then we define \( A \lor B : X \to [0,1] \) as follows:

\[
(A \lor B)(x) = \max \{A(x), B(x)\}
\]

Also, we define \( A \land B : X \to [0,1] \) as follows:

\[
(A \land B)(x) = \min \{A(x), B(x)\}
\]

By \( A \lor (A \land B) \), we mean the union (intersection) between two fuzzy sets \( A \) and \( B \) of \( X \).

**Definition 2.3:** [4] Let \( A \) be any fuzzy set in a set \( X \). The complement of \( A \), is denoted by \( 1_A - A \) or \( A^c \) and defined as follows: \( A^c(x) = 1 - A(x) \), for each \( x \in X \).

**Remark 2.4:** From definition (2.2) and definition (2.3), we have, if \( A, B \in I^X \), then \( A \lor B, A \land B \) and \( 1_X - A \in I^X \).

**Definition 2.5:** [9] A fuzzy point \( x_\lambda \) in a set \( X \) is a fuzzy set defined as follows:

\[
x_\lambda(y) = \begin{cases} 
\lambda & \text{if } y = x \\
0 & \text{otherwise,}
\end{cases}
\]

Where \( 0 < \lambda \leq 1 \). Now, \( \text{supp}(x_\lambda) = \{ y : x_\lambda(y) > 0 \} \), but
\[
x_\lambda(y) = \begin{cases} 
\lambda & \text{if } y = x \\
0 & \text{otherwise}, \text{ and } 0 < \lambda \leq 1.
\end{cases}
\]

\[\text{supp}(x_\lambda) = x, \text{ so the value at } x \text{ is } \lambda, \text{ and call the point } x \text{ its support of fuzzy point } x_\lambda \text{ and } \lambda \text{ is the height of } x_\lambda. \text{ That is, } x_\lambda \text{ has the membership degree } 0 \text{ for all } y \in X \text{ except one, say } x \in X.\]

**Definition 2.6:** [3] A fuzzy topology on a set \(X\) is a family \(T\) of fuzzy sets in \(X\) which satisfies the following conditions:

(i) \(0_X, 1_X \in T,\)

(ii) If \(A, B \in T,\) then \(A \land B \in T,\)

(iii) If \(\{A_i : i \in J\} \text{ is a family in } T,\) then \(\lor_{i \in J} A_i \in T.\)

\(T\) is called a fuzzy topology for \(X\) and the pair \((X, T)\) (or simply \(X\)) is a fuzzy topological space or fts for short. Every element of \(T\) is called \(T\)-fuzzy open set (fuzzy open set, for short). A fuzzy set is \(T\)-fuzzy closed (or simply fuzzy closed), if its complement is fuzzy open set. As ordinary topologies, the indiscrete fuzzy topology on \(X\) contains only \(0_X\) and \(1_X\) (i.e., \(\emptyset, X\)), while the discrete fuzzy topology on \(X\) contains all fuzzy sets in \(X\).

**Example 2.7:** Let \(X = [-1, 1]\), and let \(B_1, B_2\) and \(B_3\) are fuzzy sets in \(X\) defined as follows:

\[B_1(x) = \begin{cases} 
1, & \text{if } -1 \leq x < 0 \\
0, & \text{if } 0 \leq x \leq 1
\end{cases}\]

\[B_2(x) = \begin{cases} 
0, & \text{if } -1 \leq x < 0 \\
1, & \text{if } 0 \leq x \leq 1
\end{cases}\]

\[B_3(x) = \begin{cases} 
0, & \text{if } -1 \leq x < 0 \\
1/5, & \text{if } 0 \leq x \leq 1
\end{cases}\]

Let \(T = \{0_X, B_1, B_2, B_3, B_1 \lor B_3, 1_X\},\) then \(T\) is a fuzzy topology on \(X,\) and \((X, T)\) is a fts.

**Example 2.8:** Let \(X = [0, 1]\) and \(T = \{0_X, K, 1_X\}.\) Then \(T\) is a fuzzy topology on \(X,\) where \(K : X \to [0, 1]\) defined as:

\(K(x) = x^2,\) for all \(x \in X,\) and \((X, T)\) is a fts.

**Definition 2.9:** [3] Let \(A\) be any fuzzy set in a fts \(X.\) The interior of \(A\) is the union of all fuzzy open sets contained in \(A,\) denoted by \(\text{int}(A).\) That is, \(\text{int}(A) = \lor\{B : B \text{ is fuzzy open set, } B \leq A\}.\)

**Definition 2.10:** [3] Let \(A\) be any fuzzy set in a fts \(X.\) The closure of \(A\) is the intersection of all fuzzy closed sets containing \(A,\) denoted by \(\text{cl}(A).\) That is, \(\text{cl}(A) = \land\{B : B \text{ is fuzzy closed set, } B \geq A\}.\)
The most important properties of the closure and interior of fuzzy sets are listed in the following proposition.

**Proposition 2.11:**[12] If $A$ is any fuzzy set in $X$ then:
(i) $A$ is a fuzzy open (closed) set if and only if $A = \text{int}(A)$ ($A = \text{cl}(A)$),
(ii) $\text{cl}(1_X - A) = 1_X - \text{int}(A)$,
(iii) $\text{int}(1_X - A) = 1_X - \text{cl}(A)$.

**Proposition 2.12:**[13] Let $A, B$ be two fuzzy sets in a fts $X$. Then:
(i) $\text{int}(A) \leq A$, $\text{int}(\text{int}(A)) = \text{int}(A)$,
(ii) $\text{int}(A) \leq \text{int}(B)$, whenever $A \leq B$,
(iii) $\text{int}(A \wedge B) = \text{int}(A) \wedge \text{int}(B)$, $\text{int}(A \vee B) \geq \text{int}(A) \vee \text{int}(B)$,
(iv) $A \leq \text{cl}(A)$, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,
(v) $\text{cl}(A) \leq \text{cl}(B)$, whenever $A \leq B$,
(vi) $\text{cl}(A \wedge B) \leq \text{cl}(A) \wedge \text{cl}(B)$, $\text{cl}(A \vee B) = \text{cl}(A) \vee \text{cl}(B)$.

**Definition 2.13:**[1] Let $(X, T)$ be a fuzzy topological space. Then $X$ is called:
(i) fuzzy $T_0$-space ($FT_0$-space, for short) iff for each pair of distinct fuzzy points $x_\alpha$ and $y_\alpha$ of $X$, there exists an fuzzy open set in $X$ containing one and not the other.
(ii) fuzzy $T_1$-space ($FT_1$-space, for short) iff for each pair of distinct fuzzy points $x_\alpha$ and $y_\alpha$ of $X$, there exists an fuzzy open sets $G, H$ containing $x_\alpha$ and $y_\alpha$ respectively such that $y_\alpha \not\in G$ and $x_\alpha \not\in H$.
(iii) fuzzy $T_2$-space ($FT_2$-space, for short) iff for each pair of distinct fuzzy points $x_\alpha$ and $y_\alpha$ of $X$, there exist disjoint fuzzy open sets $G, H$ in $X$ such that $x_\alpha \in G$ and $y_\alpha \in H$.
(iv) fuzzy regular space iff for each fuzzy closed set $A$ and for each $x_\alpha \not\in A$, there exist disjoint fuzzy open sets $G, H$ such that $x_\alpha \in G$ and $A \subseteq H$.
(v) fuzzy normal space iff for each pair of disjoint fuzzy closed sets $A$ and $B$, there exist disjoint fuzzy open sets $G$ and $H$ such that $A \subseteq G$ and $B \subseteq H$.

3. KERNEL SET IN FUZZY TOPOLOGICAL SPACES

**Definition 3.1:** The intersection of all fuzzy open subsets of a fuzzy topological space $(X, T)$ containing $A$ is called the kernel of $A$ (briefly $\text{ker}(A)$), this means that $\text{ker}(A) = \wedge \{G \in T: A \subseteq G\}$.

**Definition 3.2:** In a fuzzy topological space $(X, T)$, a set $A$ is said to be weakly ultra separated from $B$ if there exists a fuzzy open set $G$ such that $A \subseteq G$ and $G \wedge B = 0_X$ or $A \wedge \text{cl}(B) = 0_X$.

**Remark 3.3:** By definition (3.2), we have the following: For every two distinct fuzzy points $x_\alpha$ and $y_\alpha$ of $X$, $\text{ker}\{x_\alpha\} = \{y_\alpha: \{x_\alpha\}\}$ is not weakly ultra separated from $\{y_\alpha\}$.

**Definition 3.4:** A fuzzy topological space $(X, T)$ is called fuzzy $R_0$-space ($FR_0$-space, for short) if for each fuzzy open set $U$ and $x_\alpha \in U$ then $\text{cl}\{x_\alpha\} \subseteq U$.

**Definition 3.5:** A fuzzy topological space $(X, T)$ is called fuzzy $R_1$-space ($FR_1$-space, for short) if for each two distinct fuzzy points $x_\alpha$ and $y_\alpha$ of $X$ with $\text{cl}\{x_\alpha\} \neq \text{cl}\{y_\alpha\}$, there exist disjoint fuzzy open sets $U, V$ such that $\text{cl}\{x_\alpha\} \subseteq U$ and $\text{cl}\{x_\alpha\} \subseteq V$. 
Conversely, let kernelled fuzzy point of \( g_1 \). Then \( x_\lambda \) cannot be a fuzzy point of \( X \). Thus, \( x_\lambda \) is a fuzzy point of \( X \). The converse part can be proved in a similar way.

Theorem 3.9: A fuzzy topological space \((X, T)\) is \( FT_2 \)-space if and only if for each \( x \neq y \in X \), \( y_\alpha \in ker\{x_\lambda\} \) and \( x_\lambda \in ker\{y_\alpha\} \).

Proof: Let \((X, T)\) be a \( FT_2 \)-space then for each \( x \neq y \in X \), there exists an fuzzy open sets \( U, V \) such that \( x_\lambda \in U, y_\alpha \notin U \) and \( y_\alpha \in V, x_\lambda \notin V \). Implies \( y_\alpha \notin ker\{x_\lambda\} \) and \( x_\lambda \notin ker\{y_\alpha\} \).

Conversely, let \( y_\alpha \notin ker\{x_\lambda\} \) and \( x_\lambda \notin ker\{y_\alpha\} \), for each \( x \neq y \in X \). Then there exists an fuzzy open sets \( U, V \) such that \( x_\lambda \in U, y_\alpha \notin U \) and \( y_\alpha \in V, x_\lambda \notin V \). Thus, \((X, T)\) is a \( FT_2 \)-space.

Theorem 3.10: A fuzzy topological space \((X, T)\) is \( FT_2 \)-space if and only if for each \( x \in X \) then \( ker\{x_\lambda\} = \{x_\lambda\} \).

Proof: Let \((X, T)\) be a \( FT_2 \)-space and let \( ker\{x_\lambda\} = \{x_\lambda\} \), then \( ker\{x_\lambda\} \) contains another fuzzy point distinct from \( x_\lambda \) say \( y_\alpha \).

So \( y_\alpha \notin ker\{x_\lambda\} \). Hence by theorem (3.8), \((X, T)\) is not a \( FT_2 \)-space this is a contradiction. Thus, \( ker\{x_\lambda\} = \{x_\lambda\} \).

Conversely, let \( ker\{x_\lambda\} = \{x_\lambda\} \), for each \( x \in X \) and let \((X, T)\) be not a \( FT_2 \)-space. Then, by theorem (3.8), \( y_\alpha \notin ker\{x_\lambda\} \), implies \( ker\{x_\lambda\} \neq \{x_\lambda\} \), this is a contradiction. Thus, \((X, T)\) is a \( FT_2 \)-space.

Definition 3.11: Let \((X, T)\) be a fuzzy topological space. A fuzzy point \( x_\lambda \) is said to be kernelled fuzzy point of \( A \leq X \) (Briefly \( x_\lambda \in ker(A) \)) if and only if for each \( G \) fuzzy closed set contains \( x_\lambda \) then \( G \wedge A \neq 0_X \).

Definition 3.12: Let \((X, T)\) be a fuzzy topological space. A fuzzy point \( x_\lambda \) is said to be boundary kernelled fuzzy point of \( A \) (Briefly \( x_\lambda \in ker_{bd}(A) \)) if and only if for each fuzzy closed set \( G \) contains \( x_\lambda \) then \( G \wedge A \neq 0_X \) and \( G \wedge A^c \neq 0_X \).

Definition 3.13: By definition (3.10), we have the following: For every two distinct fuzzy points \( x_\lambda \) and \( y_\alpha \) of \( X \), \( ker\{x_\lambda\} = \{y_\alpha : x_\lambda \in G_{y_\alpha}, G_{y_\alpha} \in T \} \).

Theorem 3.14: Let \((X, T)\) be a fuzzy topological space and \( x \neq y \in X \). Then \( x_\lambda \) is a kernelled fuzzy point of \( \{y_\alpha\} \) if and only if \( y_\alpha \) is a adherent fuzzy point of \( \{x_\lambda\} \).

Proof: Let \( x_\lambda \) be a kernelled fuzzy point of \( \{y_\alpha\} \). Then for every fuzzy closed set \( G \) such that \( x_\lambda \in G \) implies \( y_\alpha \in G \), then \( y_\alpha \in \wedge \{G : x_\lambda \in G \} \), this means \( y_\alpha \in cl\{x_\lambda\} \). Thus \( y_\alpha \) is an adherent fuzzy point of \( \{x_\lambda\} \).
Conversely, let $y_a$ be an adherent fuzzy point of $\{x_\lambda\}$. Then for every fuzzy open set $U$ such that $y_a \in U$ implies $x_\lambda \in U$, then $x_\lambda \in U \cap \{y_a \in U\}$, this means $x_\lambda \in \ker\{y_a\}$. Thus, $x_\lambda$ is a kernelled fuzzy point of $\{y_a\}$.

**Theorem 3.15:** Let $(X, T)$ be a fuzzy topological space and $A \subseteq X$ and let $kr_{dr}(A)$ be the set of all kernelled derived fuzzy points of $A$, then $\ker(A) = A \cup kr_{dr}(A)$.

**Proof:** Let $x_\lambda \in A \cup kr_{dr}(A)$ and if $x_\lambda \in kr_{dr}(A)$, then for every fuzzy closed set $G$ intersects $A$ (in a fuzzy point different from $x_\lambda$). Therefore, $x_\lambda \in \ker\{x_\lambda\}$. Hence, $kr_{dr}(A) \subseteq \ker(A)$, it follows that $A \cup kr_{dr}(A) \subseteq \ker(A)$. To demonstrate the reverse inclusion, we consider $x_\lambda$ be a fuzzy point of $\ker(A)$. If $x_\lambda \in A$, then $x_\lambda \in A \leq kr_{dr}(A)$. Suppose that $x_\lambda \notin A$. Since $x_\lambda \in \ker(A)$, then for every fuzzy closed set $G$ containing $x_\lambda$ implies $G \cap A \neq \emptyset$, this means $A \cap G \cap \{x_\lambda\} \neq \emptyset$. Then, $x_\lambda \in kr_{dr}(A)$, so that $x_\lambda \in A \cup kr_{dr}(A)$.

**Theorem 3.16:** Let $(X, T)$ be a fuzzy topological space and $A \subseteq X$ and let $kr_{bd}(A)$ be the set of all kernelled boundary fuzzy points of $A$, then $\ker(A) = A \cup kr_{bd}(A)$.

**Proof:** Let $x_\lambda \in A \cup kr_{bd}(A)$ and if $x_\lambda \in kr_{bd}(A)$, then for every fuzzy closed set $G$ intersects $A$, therefore, $x_\lambda \in \ker\{x_\lambda\}$. Hence, $kr_{bd}(A) \subseteq \ker(A)$, it follows that $A \cup kr_{bd}(A) \subseteq \ker(A)$. To demonstrate the reverse inclusion, we consider $x_\lambda$ be a fuzzy point of $\ker(A)$. If $x_\lambda \in A$, then $x_\lambda \in A \cup kr_{bd}(A)$. Suppose that $x_\lambda \notin A$, implies $x_\lambda \in A^c$. Since $x_\lambda \in \ker(A)$, then for every fuzzy closed set $G$ containing $x_\lambda$ implies $G \cap A \neq \emptyset$ and $G \cap A^c \neq \emptyset$. Then $x_\lambda \in kr_{bd}(A)$, so that $x_\lambda \in A \cup kr_{bd}(A)$. Hence, $\ker(A) \subseteq A \cup kr_{bd}(A)$. Thus, $\ker(A) = A \cup kr_{dr}(A)$.

**Corollary 3.17:** Every interior fuzzy point is a kernelled fuzzy point.

**Proof:** Since $\text{int}(A) \subseteq A \leq \ker(A)$. Thus, every interior fuzzy point is a kernelled fuzzy point.

**Theorem 3.18:** Let $(X, T)$ be a fuzzy topological space and $A$ is a subset of $X$. Then $A$ is a fuzzy open set if and only if every $x_\lambda$ kernelled fuzzy point of $A$ is an interior fuzzy point of $A$.

**Proof:** Let $A$ be a fuzzy open set, then $\ker(A) = A = \text{int}(A)$, implies every kernelled fuzzy point is an interior fuzzy point. Conversely, let every $x_\lambda$ kernelled fuzzy point of $A$ is an interior fuzzy point of $A$. Then $\ker(A) \leq \text{int}(A)$. Hence, $\text{int}(A) \subseteq A \leq \ker(A)$, implies $\text{int}(A) = A = \ker(A)$. Thus $A$ is a fuzzy open set.

**Corollary 3.19:** A subset $A$ of $X$ is a fuzzy open set if and only if for each $x_\lambda$ kernelled fuzzy point then $x_\lambda \notin \text{cl}(A^c)$.

**Proof:** By theorem (3.18).

**Theorem 3.20:** A subset $A$ of $X$ is a fuzzy closed set if and only if $\ker(A^c) \cap \text{cl}(A) = 0_X$.

**Proof:** Let $A$ is a fuzzy closed set. Then $A^c$ is a fuzzy open set, implies $A^c = \ker(A^c)$ by theorem (3.18). Hence $A = \text{cl}(A)$. Thus $\ker(A^c) \cap \text{cl}(A) = 0_X$. Conversely, let $\ker(A^c) \cap \text{cl}(A) = 0_X$, then for each $x_\lambda \in \ker(A^c)$, implies $x_\lambda \notin \text{cl}(A)$, implies $x_\lambda \in \text{ext}(A)$. Therefore $x_\lambda \in \text{int}(A^c)$. Hence by theorem (3.18), $A^c$ is a fuzzy open set. Thus $A$ is a fuzzy closed set.
Theorem 3.21: A fuzzy topological space \((X, T)\) is \(FR_0\)-space if and only if every adherent fuzzy point of \(\{x_\lambda\}\) is a kernelled fuzzy point of \(\{x_\lambda\}\).

**Proof:** Let \((X, T)\) be an \(FR_0\)-space. Then, for each \(x \in X, ker\{x_\lambda\} = cl\{x_\lambda\}\) by lemma (3.7). Thus, every adherent fuzzy point of \(\{x_\lambda\}\) is a kernelled fuzzy point of \(\{x_\lambda\}\).

Conversely, let every adherent fuzzy point of \(\{x_\lambda\}\) is a kernelled fuzzy point of \(\{x_\lambda\}\) and let \(U \subseteq X\) and \(x_\lambda \in U\). Then \(cl\{x_\lambda\} \leq ker\{x_\lambda\}\) for each \(x \in X\). Since \(ker\{x_\lambda\} = \{x \in U \mid U \in T, x_\lambda \in U\}\), implies \(cl\{x_\lambda\} \leq U\), for each \(U\) fuzzy open set contains \(x_\lambda\). Thus, \((X, T)\) is an \(FR_0\)-space.

Theorem 3.22: A fuzzy topological space \((X, T)\) is \(FT_0\)-space if and only if for each \(x \neq y \in X\), either \(x_\lambda\) is not kernelled fuzzy point of \(\{y_a\}\) or \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\).

**Proof:** Let \((X, T)\) be an \(FT_0\)-space. Then for each \(x \neq y \in X\) there exists a fuzzy open set \(U\) such that \(x_\lambda \in U\), \(y_a \notin U\) (say), implies \(y_a \notin U^c\). Hence \(U^c\) is a fuzzy closed, then \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\). Thus either \(x_\lambda\) is not kernelled fuzzy point of \(\{y_a\}\) or \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\).

Conversely, let for each \(x \neq y \in X\), either \(x_\lambda\) is not kernelled fuzzy point of \(\{y_a\}\) or \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\). Then there exist a fuzzy closed set \(G\) such that \(x_\lambda \in G\), \(G \wedge \{y_a\} = \emptyset\) or \(y_a \in G\), \(G \wedge \{x_\lambda\} = \emptyset\), implies \(x_\lambda \notin G^c\), \(y_a \notin G^c\) or \(x_\lambda \notin G^c\), \(y_a \notin G^c\). Hence \(G^c\) is a fuzzy open set. Thus, \((X, T)\) is an \(FT_0\)-space.

Theorem 3.23: A fuzzy topological space \((X, T)\) is \(FT_1\)-space if and only if \(kr_{dr}\{x_\lambda\} = \emptyset\), for each \(x \in X\).

**Proof:** Let \((X, T)\) be an \(FT_1\)-space. Then for each \(x \in X, ker\{x_\lambda\} = \{x_\lambda\}\) by theorem (3.9). Since \(kr_{dr}\{x_\lambda\} = ker\{x_\lambda\} - \{x_\lambda\}\). Thus, \(kr_{dr}\{x_\lambda\} = \emptyset\).

Conversely, let \(kr_{dr}\{x_\lambda\} = \emptyset\). By theorem (3.14), \(ker\{x_\lambda\} = \{x_\lambda\}\) \(\lor kr_{dr}\{x_\lambda\}\), implies \(ker\{x_\lambda\} = \{x_\lambda\}\). Hence, by theorem (3.9), \((X, T)\) is a \(FT_1\)-space.

Theorem 3.24: A fuzzy topological space \((X, T)\) is \(FT_{1-}\)space if and only if for each \(x \neq y \in X\), \(x_\lambda\) is not kernelled fuzzy point of \(\{y_a\}\) and \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\).

**Proof:** Let \((X, T)\) be a \(FT_{1-}\)space. Then for each \(x \neq y \in X\) there exist fuzzy open sets \(U, V\) such that \(x_\lambda \in U\), \(y_a \notin U\) and \(y_a \in V\), \(x_\lambda \notin V\), implies \(x_\lambda \in V^c\), \(\{y_a\} \wedge V^c = \emptyset\) and \(y_a \in U^c\), \(\{x_\lambda\} \wedge U^c = \emptyset\). Hence \(U^c\) and \(V^c\) are fuzzy closed sets. Thus \(x_\lambda\) is not kernelled fuzzy point of \(\{y_a\}\) and \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\).

Conversely, let for each \(x \neq y \in X\), \(x_\lambda\) is not kernelled fuzzy point of \(\{y_a\}\) and \(y_a\) is not kernelled fuzzy point of \(\{x_\lambda\}\). Then, there exist fuzzy closed sets \(G_1, G_2\) such that \(x_\lambda \in G_1\), \(G_1 \wedge \{y_a\} = \emptyset\) and \(y_a \in G_2\), \(G_2 \wedge \{x_\lambda\} = \emptyset\), implies \(x_\lambda \notin G_1^c\), \(y_a \notin G_2^c\) and \(y_a \in G_1^c\), \(x_\lambda \notin G_2^c\). Hence \(G_1^c\) and \(G_2^c\) are fuzzy open sets. Thus, \((X, T)\) is \(FT_{1-}\)-space.

4. kr- Spaces in Fuzzy Topological Spaces:

**Definition 4.1:** A fuzzy topological space \((X, T)\) is said to be a \(kr\)-space if and only if for each subset \(A\) of \(X\) then, \(ker(A)\) is a fuzzy open set.

**Definition 4.2:** A fuzzy topological \(kr\)-space \((X, T)\) is called fuzzy \(T_{kr}\)-space (\(FT_{kr}\)-space, for short) if and only if for each \(x_\lambda \in X\), then \(kr_{dr}\{x_\lambda\}\) is a fuzzy open set.
Theorem 4.3: In fuzzy topological $kr$-space $(X, T)$, every $FT_1$-space is a $FT_k$-space.

Proof: Let $(X, T)$ be a $FT_1$-space. Then, for each $x_\alpha \in X$, $ker\{x_\alpha\} = \{x_\alpha\}$ by theorem (3.9). As $kr_{dr}\{x_\alpha\} = ker\{x_\alpha\} - \{x_\alpha\}$, implies $kr_{dr}\{x_\alpha\} = 0_X$. Thus, $(X, T)$ is a $FT_k$-space.

Theorem 4.4: In fuzzy topological $kr$-space $(X, T)$, every $FT_k$-space is a $FT_0$-space.

Proof: Let $(X, T)$ be a $FT_k$-space and let $x \neq y \in X$. Then, $kr_{dr}\{x_\alpha\}$ is a fuzzy open set, therefore, there exist two cases:

(i) $y_\alpha \in kr_{dr}\{x_\alpha\}$ is a fuzzy open set. Since $x_\alpha \notin kr_{dr}\{x_\alpha\}$. Thus $(X, T)$ is a $FT_0$-space.

(ii) $y_\alpha \notin kr_{dr}\{x_\alpha\}$, implies $y_\alpha \notin ker\{x_\alpha\}$. But $ker\{x_\alpha\}$ is a fuzzy open set. Thus $(X, T)$ is a $FT_0$-space.

Definition 4.5: A fuzzy topological $kr$-space $(X, T)$ is said to be fuzzy $T_L$-space ($FT_L$-space, for short) if and only if for each $x \neq y \in X$, $ker\{x_\alpha\} \land ker\{y_\alpha\}$ is degenerated (empty or singleton fuzzy set).

Theorem 4.6: In fuzzy topological $kr$-space $(X, T)$, every $FT_1$-space is $FT_L$-space.

Proof: Let $(X, T)$ be a $FT_1$-space. Then for each $x \neq y \in X$, $ker\{x_\alpha\} = \{x_\alpha\}$ and $ker\{y_\alpha\} = \{y_\alpha\}$ by theorem (3.9), implies $ker\{x_\alpha\} \land ker\{y_\alpha\} = 0_X$. Thus $(X, T)$ is a $FT_L$-space.

Theorem 4.7: In fuzzy topological $kr$-space $(X, T)$, every $FT_L$-space is a $FT_0$-space.

Proof: Let $(X, T)$ be a $FT_L$-space. Then for each $x \neq y \in X$, $ker\{x_\alpha\} \land ker\{y_\alpha\}$ is degenerated (empty or singleton fuzzy set). Therefore there exist three cases:

(i) $ker\{x_\alpha\} \land ker\{y_\alpha\} = 0_X$, implies $(X, T)$ is a $FT_0$-space.

(ii) $ker\{x_\alpha\} \land ker\{y_\alpha\} = \{x_\alpha\}$ or $\{y_\alpha\}$, implies $y_\alpha \notin ker\{x_\alpha\}$ or $x_\alpha \notin ker\{y_\alpha\}$ implies $(X, T)$ is a $FT_0$-space.

(iii) $ker\{x_\alpha\} \land ker\{y_\alpha\} = \{z_\beta\}$, $z \neq x \neq y$, $z \in X$, implies $y_\alpha \notin ker\{x_\alpha\}$ and $x_\alpha \notin ker\{y_\alpha\}$, implies $(X, T)$ is a $FT_0$-space.

Definition 4.8: A fuzzy topological $kr$-space $(X, T)$ is said to be a fuzzy $T_N$-space ($FT_N$-space, for short) if and only if for each $x \neq y \in X$, $ker\{x_\alpha\} \land ker\{y_\alpha\}$ is empty or {$x_\alpha$} or {$y_\alpha$}.

Theorem 4.9: In fuzzy topological $kr$-space $(X, T)$, every $FT_1$-space is $FT_N$-space.

Proof: Let $(X, T)$ be a $FT_N$-space. Then for each $x \neq y \in X$, $ker\{x_\alpha\} = \{x_\alpha\}$ and $ker\{y_\alpha\} = \{y_\alpha\}$ by theorem (3.9), implies $ker\{x_\alpha\} \land ker\{y_\alpha\} = 0_X$. Thus $(X, T)$ is a $FT_N$-space.

Theorem 4.10: In fuzzy topological $kr$-space $(X, T)$, every $FT_N$-space is a $FT_0$-space.

Proof: Let $(X, T)$ be a $FT_N$-space. Then for each $x \neq y \in X$, $ker\{x_\alpha\} \land ker\{y_\alpha\}$ is degenerated (empty or singleton fuzzy set). Therefore there exist two cases:

(i) $ker\{x_\alpha\} \land ker\{y_\alpha\} = 0_X$, implies $(X, T)$ is a $FT_0$-space.

(ii) $ker\{x_\alpha\} \land ker\{y_\alpha\} = \{x_\alpha\}$ or $\{y_\alpha\}$, implies $y_\alpha \notin ker\{x_\alpha\}$ or $x_\alpha \notin ker\{y_\alpha\}$, implies $(X, T)$ is a $FT_0$-space.
Theorem 4.11: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_2 \)-space iff for each \( x \neq y \in X \), then \( ker\{x \} \land ker\{y\} = 0_X \).

Proof: Let a fuzzy topological \( kr \)-space \((X, T)\) is \( FT_2 \)-space. Then for each \( x \neq y \in X \) there exist disjoint fuzzy open sets \( U, V \) such that \( x \in U \), and \( y \in V \). Hence \( ker\{x\} \leq U \) and \( ker\{y\} \leq V \). Thus \( ker\{x\} \land ker\{y\} = 0_X \).

Conversely, let for each \( x \neq y \in X \), \( ker\{x\} \land ker\{y\} = 0_X \). Since \((X, T)\) is a fuzzy topological \( kr \)-space, this means kernel is a fuzzy open set. Thus \((X, T)\) is \( FT_2 \)-space.

Theorem 4.12: A fuzzy topological \( kr \)-space \((X, T)\) is fuzzy regular space iff for each \( G \) fuzzy closed set and \( x_1 \notin G \), then \( ker(G) \land ker\{x_1\} = 0_X \).

Proof: By the same way of proof of theorem (4.11).

Theorem 4.13: A fuzzy topological \( kr \)-space \((X, T)\) is fuzzy normal space iff for each disjoint fuzzy closed sets \( G, H \), then \( ker(G) \land ker(H) = 0_X \).

Proof: By the same way of proof of theorem (4.11).

Theorem 4.14: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_1 \)-space iff it is \( FR_0 \)-space and \( FT_k \)-space.

Proof: It follows from theorem (4.3) and remark (3.6).

Theorem 4.15: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_1 \)-space iff it is \( FR_0 \)-space and \( FT_L \)-space.

Proof: It follows from theorem (4.6) and remark (3.6).

Theorem 4.16: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_1 \)-space if and only if it is \( FR_0 \)-space and \( FT_N \)-space.

Proof: It follows from theorem (4.9) and remark (3.6).

Theorem 4.17: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_i \)-space if and only if it is \( FR_{i-1} \)-space and \( FT_k \)-space, \( i = 1, 2 \).

Proof: It follows from theorem (4.3) and remark (3.6).

Theorem 4.18: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_i \)-space if and only if it is \( FR_{i-1} \)-space and \( FT_L \)-space, \( i = 1, 2 \).

Proof: It follows from theorem (4.6) and remark (3.6).

Theorem 4.19: A fuzzy topological \( kr \)-space \((X, T)\) is \( FT_i \)-space if and only if it is \( FR_{i-1} \)-space and \( FT_N \)-space, \( i = 1, 2 \).

Proof: It follows from theorem (4.9) and remark (3.6).
**Remark 4.20:** The relation between fuzzy separation axioms can be representing as a matrix. Therefore, the element \( a_{ij} \) refers to this relation. As the following matrix representation shows:

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**Matrix Representation**

The relation between fuzzy separation axioms in fuzzy topological \( kr \)-spaces

**REFERENCES**