ANALYSIS OF BALL BEARING THROUGH FEM

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ABSTRACT

This article is all about the critical problem occur in ball bearing rolling element bearing are vital components in almost every machines or applications with moving or rotating parts therefore it is great importance to fully understand the behaviour and the nature of the bearing. Now adays there are good ways to predict the expected life of a rolling element bearing but there is no way to calculate the exact service life and which can lead to serious and costly consequences in the event of breakdown of the bearing. Since for the diagnosis of ball bearing faults which comes through vibrations. With proper knowledge and diagnostic procedure it is normally possible to quickly pinpoint the vibrations. This may result in trying to fix an incorrectly diagnosis problem, spending a significant amount of time and money in the process. by utilizing the proper data collection and analysis techniques, the true source of the vibrations can be discovered.

Keywords: Ball Bearing, Guyanreduction, DOF(Degree of freedom).

INTRODUCTION

In this article we are dealing with the concept of non linear vibration absorber occurs in ball bearing. The key feature of this type of absorber is their lack of preferential frequencies that is they also can reduce vibration at virtually in any frequency. This provides a great advantage over the classical linear absorber which are typically limited to a single frequency. The most important downside is their amplitude dependency that is they only succeed in achieving efficient vibration reduction in a well defined amplitude region. Recently, an inventive and very interesting approach has opened new doors towards the development of a frequency-robust mechanical vibration controlling device. The novelty is to replace the linear spring in the conventional tuned mass damper by a strongly nonlinear one, where strongly nonlinear refers to a nonlinearizable spring. By introducing the nonlinear spring, a much wider variety of sometimes very complicated response regimes is possible. The challenge lies in picking out and controlling the desired one. The idea of
using a strongly nonlinear spring absorber was first proposed by Gendelman. In his case, the desired response regime was closely related to an energy pumping phenomenon, i.e. the one-way energy transfer from the linear primary system to the nonlinear attachment, which is of course interesting when it comes to vibration reduction. Besides the pumping phenomenon, two other key concepts were addressed in the same article. The first one is the existence of a critical energy threshold below which no efficient vibration reduction is possible. In other words, the energy imparted into the primary system must be high enough for the absorber to work properly, a crucial feature regarding the tuning procedure of the absorber. The second concept is that a strongly nonlinear absorber does not feature a preferential frequency which allows it to work as a multi-frequency vibration reduction element. The concepts outlined by Gendelman have been further developed and analyzed throughout the years. Most of the results are gathered in where the focus lies on the energy pumping phenomenon or targeted energy transfer (TET) from the linear primary system to the nonlinear vibration absorber. Much effort is put in understanding the transient dynamics (which appear to be very complicated) of the following seemingly simple 2DOF system

\[
\begin{align*}
(m\ddot{x} + kx + cna(\dot{x} - \dot{x}_{na}) + kna(x - x_{na})^3 &= 0 \\
ma\ddot{x}_{na} + cna(\dot{x}_{na} - \dot{x}) + kna(x_{na} - x)^3 &= 0.
\end{align*}
\]

Despite its complicated behavior, the nonlinear attachment succeeds in achieving a broadband vibration reduction. Compared to its linear counterpart, the strongly nonlinear vibration absorber forms a relatively new research field, many aspects of which are still to be explored.

![Diagram of Damped nonlinear vibration absorber attached to undamped SDOF primary system](image)

In above Figure Damped nonlinear vibration absorber attached to undamped SDOF primary system. In this dissertation many different aspects of vibration absorbers are investigated. Although a great deal of work has already been performed, we are convinced that the current work succeeds in adding important insights which contribute to a successful practical implementation of the vibration absorber. The thesis is subdivided into three parts. Part I deals with the linear vibration absorber attached to a MDOF system under harmonic load, Part II with the nonlinear vibration absorber attached to a SDOF system undergoing transient excitation. Part III exemplifies the use of vibration absorbers in a real-life industrial application.

A relatively new element in passive vibration control is the strongly nonlinear vibration absorber, i.e. it features a non linearizable stiffness characteristic. The key feature of these type of absorbers is their lack of a preferential frequency, i.e. they can reduce vibrations at virtually any frequency. This provides a great advantage over the classical linear absorbers which are typically limited to a single frequency. The most important downside is their amplitude dependency, i.e. they
only succeed in achieving efficient vibration reduction in a well defined vibration amplitude region. More precisely, a critical energy threshold exists which means that efficient vibration reduction is only possible above a minimum amount of input energy into the primary system. Because this defines a crucial concept in the design procedure of nonlinear vibration absorbers, we have studied it into detail. The main contribution is a dimensionless approach which enables to determine the boundaries between weak and strong vibration reduction for a wide variety of systems. For a cubic nonlinear vibration absorber attached to a linear oscillator (LO): Subject to initial conditions, the equations of motion become

\[
\begin{align*}
    m\ddot{x} + c\dot{x} + kx + c_{na}(\dot{x} - \dot{x}_{na}) + k_{na}(x - x_{na})^3 &= 0 \\
m_{na}\ddot{x}_{na} + c_{na}(\dot{x}_{na} - \dot{x}) + k_{na}(x_{na} - x)^3 &= 0
\end{align*}
\]  

(9)

where \(m\), \(c\) and \(k\) denote the mass, damping and stiffness of the linear oscillator, while subscript ‘\(na\)’ is used for the nonlinear absorber. The existence of the energy threshold is evidenced by in above Fig. where a small increase in the LO’s initial velocity, causes a large increase in the absorber’s response and its ability to dissipate vibrational energy.

In above figure: Main system and cubic absorber response (\(m = 1, c = 0, k = 1, m_{na} = 0.05, c_{na} = 0.016, k_{na} = 0.067\)); blabla (a) \(x(0) = 0.5, x(0) = \dot{x}_{na}(0) = x_{na}(0) = 0\) bl (b) \(x(0) = 0.6, x(0) = \dot{x}_{na}(0) = x_{na}(0) = 0\)

To improve insight in this remarkable behavior, an approximate theoretical analysis has been performed. The key approximation is neglecting terms \(m_{na}/m \ll 1\) which allows to reduce the initial 2DOF system of (9) to a SDOF system in dimensionless form:

\[
\frac{d^2 x_d}{d\tau^2} + \gamma(x_d)^3 = \sin(\tau + \phi)
\]  

(10)

with initial conditions

\[
\begin{align*}
    x_d(0) &= \frac{(x_{na}(0) - x(0))\omega_0^2}{\sqrt{\omega_0^2 + \omega_0^2}} \\
    \dot{x}_d(0) &= \frac{(x_{na}(0) - x(0))\omega_0}{\sqrt{\omega_0^2 + \omega_0^2}}
\end{align*}
\]  

(11)

The main advantage of (10) is that \(k_{na}, m_{na}\), the LO’s natural frequency \(\omega_0\), \(x(0)\) and \(\dot{x}(0)\) are joined into one dimensionless parameter \(\gamma > 0\)

\[
\gamma = \frac{k_{na}}{m_{na}} \frac{(x(0))^2 + (\dot{x}(0))^2}{\omega_0^2}.
\]  

(12)

Inertia of the Absorber Stiffness of the Absorber (steel wire) Inertia of the Primary system Stiffness of the Primary system (spring steel shaft) Motor

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In above Figure Torsional vibration setup - Primary system and absorber Using $\gamma$ as the bifurcation parameter in, we have shown that the initial energy threshold value, is transformed into a threshold value for $\gamma$

$$\gamma = \frac{k_{na}}{m_{na}} \left( x(0)^2 \omega^2_0 + \dot{x}(0)^2 \right) > \gamma_c \approx 0.18$$

As a result, the energy threshold concept is extended to a more general parameter threshold where each parameter (including the initial energy) is featuring its own threshold value. The same approach can be extended to cope with general nonlinear spring characteristics, nonlinear main systems and forced LO.

Inertia of the Absorber, Stiffness of the Absorber (steel wires)
Inertia of the Primary system, Stiffness of the Primary system (spring steel shaft)

In above figure Torsional vibration setup-primary system and absorber
In above figure Primary system and absorber response to a motor impulse - below (left) and above (right) energy threshold

In above Figure Steady state response of the primary system and the absorber - below (left) and above (right) energy threshold

**Part III: Towards an industrial application**-The last part of this dissertation is concerned with the feasibility of using vibration absorbers in an industrial application. An important step towards the design of a vibration absorber is reducing the generally complex FEM model of the primary system into a system of manageable size able to represent the low frequency behaviour. A very popular technique used in many software packages is the Guyan reduction technique where the entire set of coordinates is divided into master coordinates $q_m$ and slave coordinates.

\[
\begin{bmatrix}
M_{mm} & M_{ms} \\
M_{sm} & M_{ss}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_m \\
\dot{q}_s
\end{bmatrix}
+ \begin{bmatrix}
K_{mm} & K_{ms} \\
K_{sm} & K_{ss}
\end{bmatrix}
\begin{bmatrix}
q_m \\
q_s
\end{bmatrix}
= \begin{bmatrix}
f_m \\
0
\end{bmatrix},
\]

(14)

The slave coordinates are assumed to have a negligible effect on the low frequency behavior such that they can be eliminated by applying the following transformation

\[
M_m = S_f^T M_s S_f \\
K_m = S_f^T K_s S_f
\]

(15)

with

\[
\begin{bmatrix}
q_m \\
q_s
\end{bmatrix}
= \begin{bmatrix}
I \\
-K_s^{-1} K_{sm}
\end{bmatrix}
\begin{bmatrix}
q_m \\
q_s
\end{bmatrix}
= S_f \begin{bmatrix}
q_m
\end{bmatrix}.
\]

(16)

A crucial step is the selection of the slave coordinates. A widely implemented approach is the numerical algorithm where the slave DOFs are condensed out one by one leaving only the master coordinates. The corresponding degree of freedom is selected as a slave coordinate and condensed out by Guyan reduction, transforming $K_n$ and $M_n$ into $n - 1$ by $n - 1$ matrices. The cycle then
recommences by selecting the maximum $k_{ii}/m_{ii}$ of the reduced $K_{n-1}$ and $M_{n-1}$ matrices. In this way, the slave degrees of freedom are eliminated one by one leading to a set of master degrees of freedom. To overcome this problem, a new algorithm is proposed which sequentially selects the master DOFs, i.e. a sequential selection method (SSM). Moreover, both methods are based on the exact same principles resulting in a comparable accuracy. The developed algorithm has been successively applied in an industrial environment, namely the drive line of a hopper dredger which consists of a diesel engine driving a dredge pump, a propeller and a generator. In between these main elements are flexible couplings and gearboxes which suffer from troublesome torsional vibrations. Although these torsional vibrations also arise from steady state excitation.

![Multimodal response at the gearbox with linear absorber](image1)

**Figure 16:** Multimodal response at the gearbox with linear absorber ($m_a = 200$ kg m$^2$)

![Multimodal response at the gearbox with nonlinear absorber](image2)

**Figure 17:** Multimodal response at the gearbox with nonlinear absorber ($m_{na} = 200$ kg m$^2$)

(e.g. from the diesel engine), only transient loads are considered because the hopper dredger is, more than any other ship, subject to highly frequent changes both in configuration and load. Many of these transient loads initiate a multimodal response which calls for the application of a strongly nonlinear vibration absorber. The parameters of the nonlinear absorber (with cubic nonlinearity) are determined from Part II. Using the same absorber mass demonstrate that the cubic spring absorber outperforms the classical linear absorber. In the initial phase both absorbers succeed in reducing the energy of one vibration mode which rapidly decreases due to a beating phenomenon. After this initial phase, the linear absorber becomes ineffective. The nonlinear absorber however, succeeds in reducing the energy of the remaining vibration modes as well. This happens in a sequential manner through so called
resonance capture cascades which unfortunately exhibit a much slower vibration reduction. Further increasing the absorber mass improves this situation (Fig. 18), but the question remains whether such a large mass is practically implementable. For a detailed feasibility study one should also take into account the excitation of the diesel engine, the speed governor and the primary system’s damping.

Consider the equations of motion of an undamped model with \( n \) degrees of freedom

\[
M_n \ddot{q}_n + K_n q_n = f_n
\]

Where \( M_n \in \mathbb{R}^{n \times n} \) and \( K_n \in \mathbb{R}^{n \times n} \) are the mass and stiffness matrices and \( f_n \in \mathbb{R}^{n \times 1} \) is the force vector. The degrees of freedom of the full model are subdivided into master and slave degrees of freedom (subscripts \( m \) and \( s \) respectively above Eq can be rewritten as follows….

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS**

\[
\begin{bmatrix}
M_{mm} & M_{ms} \\
M_{sm} & M_{ss}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_m \\
\dot{q}_s
\end{bmatrix}
+
\begin{bmatrix}
K_{mm} & K_{ms} \\
K_{sm} & K_{ss}
\end{bmatrix}
\begin{bmatrix}
q_m \\
q_s
\end{bmatrix}
=
\begin{bmatrix}
f_m \\
0
\end{bmatrix}.
\]

(7.2)

Static reduction implies that the inertia forces of the slave degrees of freedom given in Eq. (7.2) are neglected

\[
M_{sm} \ddot{q}_m + M_{ss} \ddot{q}_s = 0.
\]

(7.3)

In this way, the slave degrees of freedom \( q_s \) can be eliminated or condensed out by considering their deformation as a static function of the deformation of the master degrees of freedom \( q_m \)

\[
K_{sm} q_m + K_{ss} q_s = 0 \quad \Rightarrow \quad q_s = S_d q_m
\]

(7.4)

where \( S_d = -K_{ss}^{-1} K_{sm} \). A reduced model can now be obtained with mass and stiffness matrices \( M_m, K_m \in \mathbb{R}^{m \times m} \)

\[
M_m = S_f^T M_n S_f \quad K_m = S_f^T K_n S_f
\]

(7.5)

where \( m < n \) is the number of master degrees of freedom. The transformation matrix \( S_f \) describes the static transformation between the degrees of freedom of the full model and the master degrees of freedom

\[
\begin{bmatrix}
q_m \\
q_s
\end{bmatrix}
= 
\begin{bmatrix}
I \\
-K_{ss}^{-1} K_{sm}
\end{bmatrix}
\begin{bmatrix}
q_m
\end{bmatrix}
= S_f \begin{bmatrix}
q_m
\end{bmatrix}.
\]

(7.6)
Since the slave degrees of freedom are condensed out using the static deformation, the reduced model is only correct at frequency zero. As the frequency increases, the dynamic forces become more significant, decreasing the accuracy of the reduced model.

**SELECTION OF MASTER DEGREES OF FREEDOM**

This method starts by selecting the maximum $k_{ii}/m_{ii}$ where $k_{ii}$ and $m_{ii}$ are diagonal entries of the full $n \times n$ $K_n$ and $M_n$ matrices. The corresponding degree of freedom is condensed out by Guyan reduction, transforming $K_n$ and $M_n$ into $(n-1) \times (n-1)$ matrices. The cycle then recommences by selecting the maximum $k_{ii}/m_{ii}$ of the reduced $K_{n-1}$ and $M_{n-1}$ matrices.

The reasoning behind this method is that a high value of $k_{ii}/m_{ii}$ will not influence the lower frequency dynamics of interest. The number of iterations in this procedure and hence the amount of master degrees of freedom required is addressed by [59] where a cut-off frequency $\omega_c$ is introduced.

The process outlined above is repeated until the maximum $k_{ii}/m_{ii}$ is smaller than $\omega_c$. As the number of slave degrees of freedom is much higher than the remaining master degrees of freedom, the question arises whether it is possible to go the other way around, i.e. selecting one by one the master degrees of freedom instead of the slave degrees of freedom.

**SEQUENTIAL SELECTION METHOD (SSM)**

Three different types of SSM are presented: a straightforward expansion of SEM based on the minimum value of $k_{ii}/m_{ii}$, a selection method based on energy considerations, a new method based on the Guyan reduction technique itself which forms the main contribution.

With a similar reasoning, the degree of freedom corresponding to the minimum $k_{ii}/m_{ii}$ could be selected as a master degree of freedom because it has a high influence on the lower frequency dynamics. Unfortunately, this proposition fails for two reasons:

1. In SEM the effect of an eliminated slave degree of freedom is redistributed to the remaining degrees of freedom by applying Guyan reduction in each iteration step as stated by. As opposed to SEM, the straightforward expansion is not an iterative procedure. Consequently, the effect of selecting a master degree of freedom is not taken into account in the selection of the remaining ones, thereby reducing the accuracy of the selection procedure.
2. Consider the longitudinal vibration of the spring-mass system.

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS 7-5**

![Figure: Upper: Spring mass structure with 4 DOFs; Lower: Spring mass structure with small m and large k inserted](image)

**Figure:** Upper: Spring mass structure with 4 DOFs; Lower: Spring mass structure with small m and large k inserted.
Clearly, m4 should be selected as a master degree of freedom because it dominates the low frequency behaviour. As the minimum $k_{ii}/m_{ii}$ is equal to $k_{3}/m_{4}$, the straightforward expansion of SEM succeeds in selecting the correct master. However, when a very small mass $m$ is added and connected to m4 by a very large stiffness $k$, as shown in the lower part of above Fig, the method falls short. Indeed, $k$ and $m$ can be determined such that:

$$\frac{k_1}{m_1} \approx \frac{k_1 + k_2}{m_2} \approx \frac{k_2 + k_3}{m_3} \leq \frac{k}{m_4} \leq \frac{k + k_3}{m}$$  \hspace{1cm} (7.8)$$

where $m_1$, $m_2$ or $m_3$ will be erroneously selected as masters rather than m4. Indeed, m4 still dominates the lower frequency dynamics because in the limit $k \to \infty$ and $m \to 0$, adding $m$ and $k$ does not alter the dynamics at all. Energy based selection method. This method proposed by selects the master degrees of freedom based on the energy distribution matrix $G$:

$$G = \Phi \otimes M_n\Phi \quad \text{or} \quad G(i, j) = \sum_{k=1}^{n} \Phi(i, k)M_n(i, k)\Phi(j, k) \quad i = 1...n, j = 1...n$$ \hspace{1cm} (7.9)$$

where $\Phi \in \mathbb{R}^{n \times n}$ is the mass normalized modal matrix and where $\otimes$ represents the cross multiplication of matrices:

$$C = A \otimes B \quad \text{or} \quad C(i, j) = A(i, j)B(i, j) \quad i = 1...n, j = 1...n$$ \hspace{1cm} (7.10)$$

Each row of $G$ corresponds to a degree of freedom and indicates the relative contribution of the different vibration modes to the total energy of that degree of freedom. Each column on the other hand corresponds to a certain vibration mode.

$$\sum_{i=1}^{n} G(i, j) = 1 \quad j = 1...n \quad \text{and} \quad \sum_{j=1}^{n} G(i, j) = 1 \quad i = 1...n$$  \hspace{1cm} (7.11)$$

Consequently, a partial row sum given by

$$\sum_{j=1}^{l} G(i, j) \quad (l < n) \quad i = 1...n$$  \hspace{1cm} (7.12)$$

It can be used to determine the relative importance of the different degrees of freedom in the first l vibration modes. Hence, if a reduced model is required with an accurate estimate up to vibration mode l, the degrees of freedom with the highest partial row sums $\sum_{j=1}^{n} G(i, j), i = 1...n$, are selected as masters. The accuracy of each vibration mode is determined according to the energy that remains in the $n - m$ degrees of freedom that were not selected as masters

$$\sum_{i=1}^{n-m} G(i, j) \quad (m < n) \quad j = 1...l$$  \hspace{1cm} (7.13)$$

Although this energy approach has an interesting basic principle, making an accurate selection of the master degrees of freedom promising. To determine the energy distribution matrix, the eigenvectors are required. Because these are not known a priori, Ritz vectors are calculated which unfortunately tend to contain higher modes. To put more emphasis on the lower modes, a weighting factor $w_{ij}$ is introduced in when selecting the masters DOFs.
\[ \sum_{j=1}^{l} C^l(i, j) w_j \quad (l < n) \quad i = 1 \ldots n \quad (7.14) \]

\[ w_j \approx 1 - \frac{\lambda_j}{\lambda_{l+1}} \quad j = 1 \ldots l \quad (7.15) \]

where \( \lambda_j \) is an eigenvalue of the reduced system governed by \( K_m \) and \( M_m \). The choice of \( w_j \) is however questionable because it is based on an eigenvalue error which is assumed to be equivalent with an error in the energy distribution matrix. Furthermore, after selecting the master degrees of freedom by means of energy considerations, the reduced model is obtained through Guyan reduction which does not imply the same energy considerations.

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS 7-7**

Therefore, isn’t it possible to select the master degrees of freedom based on the Guyan reduction principle itself? This question is answered in the next section where the main contribution is outlined: a sequential selection method (SSM) based on Guyan reduction itself.

### 7.4.1 The basic principle

**Case 1:** \( M_{ms} = M_{sm} = 0 \). Consider the \( n \) degree of freedom system of Eq. (7.2) with \( M_{ms} = M_{sm} = 0 \).

\[
\begin{bmatrix} M_{mm} & 0 \\ 0 & M_{ss} \end{bmatrix} \begin{bmatrix} q_m \\ q_s \end{bmatrix} + \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} q_m \\ q_s \end{bmatrix} = \begin{bmatrix} f_m \\ 0 \end{bmatrix} \quad (7.16)
\]

According to the second part of Eq. (7.16), the slave degrees of freedom are related to the master degrees of freedom as follows:

\[ M_{ss} q_s + K_{sm} q_m + K_{ss} q_s = 0 \quad (7.17) \]

In the frequency domain, Eq. (7.17) can be rewritten as:

\[ Q_s(\omega) = -[K_{ss} - M_{ss}\omega^2]^{-1}K_{sm}Q_m(\omega) \quad (7.18) \]

where \( Q_s \) and \( Q_m \) denote the fourier transforms of \( q_m \) and \( q_s \) respectively. In Guyan reduction, the inertia forces of the slave degrees of freedom are neglected \( (M_{ss}q_s = 0) \), such that \( Q_s \) in Eq. (7.18) is estimated by \( Q_s(\omega) \) as follows:

\[ \hat{Q}_s(\omega) = -[K_{ss}]^{-1}K_{sm}Q_m(\omega) \quad (7.19) \]

assuming that \( K_{ss} \) is not singular. By manipulating Eq. (7.19), the difference between Eqs. (7.18) and (7.19) can be determined in closed form:

\[
\hat{Q}_s(\omega) = -[K_{ss}]^{-1}K_{sm}Q_m(\omega) - \left[ I_s - K_{ss}^{-1}M_{ss}\omega^2 \right] \left( [K_{ss} - M_{ss}\omega^2]^{-1}K_{sm}Q_m(\omega) \right) \\
= -I_s + K_{ss}^{-1}M_{ss}\omega^2 \left( [K_{ss} - M_{ss}\omega^2]^{-1}K_{sm}Q_m(\omega) \right) \\
= \frac{I_s - K_{ss}^{-1}M_{ss}\omega^2}{Error} \left( Q_s(\omega) \right) 
\]

with \( I_s \in \mathbb{R}^{s \times s} \) the identity matrix. Equation (7.23) shows that the error between the real \( Q_s \) and the Guyan reduced \( \hat{Q}_s \) is equal to \( K_{ss}^{-1}M_{ss}\omega^2 \). Based on this error, which is further discussed in section 7.4.2, the master degrees of freedom can be selected.
Case 2: $M_{sm} \neq 0$ and $M_{ms} \neq 0$. Equation (7.18) changes into:

$$Q_s(\omega) = -[K_{ss} - M_{ss}\omega^2]^{-1}[K_{sm} - M_{sm}\omega^2]Q_m(\omega) \quad (7.24)$$

In this case, the error introduced by Guyan reduction cannot be determined in closed form. However, assuming from now on that element-wise $|M_{sm}(i,j)\omega^2| < |K_{sm}(i,j)|$, reduces Eq. (7.24) to (7.18) such that the error can still be approximated by $K_{ss}^{-1}M_{ss}\omega^2$.

The basic problem is to determine which of the slave degrees of freedom in $Q_s$ causes the largest error when applying Guyan reduction and should consequently become a master degree of freedom. The solution is however not straightforward because this error is influenced by all slave degrees of freedom $Q_{si}$, $i = 1...s$. Therefore, we propose an approximate selection method in which only the diagonal elements of $K^{-1}ssM_{ss}\omega^2$ are considered. This approach proves to be justifiable in many engineering applications. In this case, a true relative error on component level can be determined according to Eq. (7.25):

$$\frac{\dot{Q}_{si}(\omega) - Q_{si}(\omega)}{Q_{si}(\omega)} = (K_{ss}^{-1}M_{ss}\omega^2)(i,i) \quad (7.26)$$

Consequently, the master coordinate can be selected according to the maximum diagonal element $(K^{-1}ssM_{ss})(i,i)$. This result is used in the selection procedure.

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS 7-9**

2. In the classical selection procedure, proposed to determine the number of eliminated slaves according to $\omega_c$, a cut-off frequency which is approximately three times larger than the largest frequency of interest $\omega_{max}$. Slave degrees of freedom are condensed out as long as $|Kss(i,i)/Mss(i,i)| > \omega^2 c$. According to Eq. (7.29), this means that the maximum allowed relative error is given by:

$$\left|\frac{Q_{si}(\omega_{max}) - Q_{si}(\omega_{max})}{Q_{si}(\omega_{max})}\right| = \frac{|M_{ss}(i,i)\omega_{max}^2|}{K_{ss}(i,i)} \quad (7.30)$$

A similar approach is implemented in the selection procedure of the new sequential selection method described. Master degrees of freedom are selected until:
In a first step all the degrees of freedom are considered to be slave degrees of freedom such that the initial \( K_{ss} \) and \( M_{ss} \) are equal to \( K_n \) and \( M_n \) of the full model. The master degrees of freedom are then selected one by one according to the maximum element \( |K^{-1}_{ss} M_{ss}(i,i)| \) as explained. After each master degree of freedom selection, \( K_{ss} \) and \( M_{ss} \) are recomputed.

1. Select \( K_n \) and \( M_n \) of the full model as initial \( K_{ss} \) and \( M_{ss} \) matrices.
2. Select the degree of freedom, corresponding to the maximum element \( |K^{-1}_{ss} M_{ss}(i,i)| \), as a master.
3. Reorganize \( K_n \) and \( M_n \) according to obtain the new \( K_{ss} \) and \( M_{ss} \) matrices.
4. Repeat steps 2 and 3 until \( |(K^{-1}_{ss} M_{ss})(i,i)|\omega_{2\text{max}} < 0.1 \)

Systems with rigid body modes

For systems with a singular \( K_n \) matrix (systems with rigid body modes), an adapted version of the selection procedure is required because in this case \( K_{ss} \) is initially singular. For systems with only one rigid body mode. This problem can be circumvented by introducing a virtual master according to the maximum \( k_{ii}/m_{ii} \), where \( k_{ii} \) and \( m_{ii} \) are diagonal entries of \( K_n \) and \( M_n \). After reorganizing \( K_n \) and \( M_n \) according to Eq. (7.2), the first real master is determined according to the maximum \( |K^{-1}_{ss} M_{ss}(i,i)| \), where \( K_{ss} \) and \( M_{ss} \) are nonsingular matrices of dimension \( n - 1 \times n - 1 \). For systems with multiple rigid body modes, multiple virtual masters could be introduced until \( K_{ss} \) becomes nonsingular. Another way to avoid the problem of rigid body modes is by transforming \( M_n \) and \( K_n \) by elementary operations such that \( K_n \) contains as much zero rows as there are rigid body modes. Consequently, the remaining submatrix of \( K_n \) corresponds to the elastic motion of the structure and is nonsingular which means that the selection procedure can be applied on this submatrix. In this technique, \( \psi M_n \) \((\psi > 0)\) is added to the original stiffness matrix \( K_n \), changing the eigenfrequencies \( \omega_2 \) of the original system into \( \omega_2 + \psi \)

\[
\begin{align*}
K_n \phi &= \omega^2 M_n \phi \\
(K_n + \psi M_n) \phi &= \omega^2 M_n \phi + \psi M_n \phi \\
K'_n \phi &= \omega'^2 M_n \phi
\end{align*}
\]

with \( K_0 n = K_n + \psi M_n \) and \( \omega 02 = \omega 2 + \psi \). The value of \( \psi \) must be large enough to avoid numerical problems, but small enough to guarantee the accuracy of the selection procedure, i.e. \( \psi << \omega 1 \) with \( \omega 1 \) the frequency of the first elastic mode.

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS 7-11**

7.5.1 Computational efficiency: full \( K_n, M_n \)

The computational complexity of both methods can be compared according to the number of multiplications. The classical approach involves \( s = n - m \) reduction steps and the calculation of \( K_{m}(i,i)/M_{m}(i,i) \) in each step. The new approach on the other hand involves \( m \) steps (instead of \( n-m \)) and the calculation of \( \operatorname{K}^{-1} ss \ M_{ss} \) in each step. After this procedure, \( K_n \) and \( M_n \) are reduced in one step as opposed to the step by step reduction in the classical approach. The required number of multiplications for the step by step Guyan reduction (without calculation of \( K_{m}(i,i)/M_{m}(i,i) \) in each step) can be obtained by rewriting \( K_m = ST f K_n S_f \) and \( M_m = ST f M_n S_f \) as given by Eq. (7.5):
\[
K_m = K_{mm} - \frac{K_{ms}K_{ss}^{-1}K_{sm}}{m(m+1)/2 + m}
\]
(7.37)

\[
M_m = M_{mm} + \frac{K_{ms}K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sm} - K_{ms}K_{ss}^{-1}K_{sm} - K_{ms}K_{ss}^{-1}M_{sm}}{m^2 + m}
\]
(7.38)

with \(K_{ss}\) and \(M_{ss}\) scalars, yielding the following total number of multiplications:

\[
\sum_{p=m}^{n-1} 2p^2 + 4p + 2
\]
(7.39)

For the one step Guyan reduction (without the calculation of \(K_{ss}^{-1}M_{ss}\) in each step), Eq. (7.5) becomes:

\[
K_m = K_{mm} + \frac{K_{ms}S_d}{m^2 + s}
\]
(7.40)

\[
M_m = M_{mm} + \frac{S_d^2 M_{ss}S_d + M_{ms}S_d + S_d^2 M_{sm}}{m^2 s + (m(s + 1))}
\]
(7.41)

with \(S_d \in \mathbb{R}^{s \times m}\). The calculation of \(S_d = -K_{ss}^{-1}K_{sm}\) by applying Gaussian elimination requires \((s^3 - s)/3 + ms^2\) multiplications [63] and brings the total number of multiplications for this one step Guyan reduction to:

\[
\frac{s^3}{3} + s^2 \left(\frac{m^2}{2} + \frac{3mt}{2}\right) + s \left(\frac{5m^2}{2} + m - 1\right)
\]
(7.42)

Table presents a comparison between the multiplications required by the step by step Guyan reduction and its block form alternative given by Eqs.(7.39,7.42) respectively. For a low number of masters \((m < 103)\) the same order of multiplications is obtained. When the number of masters increases beyond \(m = 103\), much more multiplications are required when implementing Guyan reduction in block form (one step). Besides the implementation of Guyan reduction itself, both methods also require extra calculations for the selections of the master DOFs. In the classical approach, each step involves \(m\) multiplications for calculating \(K_{m(i,i)}/M_{m(i,i)}\) bringing the total number of multiplications for the entire procedure to:

\[
\sum_{p=m}^{n-1} 2p^2 + 5p + 2
\]

The new approach on the other hand requires the calculation of $K^{−1}ss M^{ss}$ in each step. By applying Gaussian elimination this yields for $m$ steps:

$$\sum_{p=n-m+1}^{n} \frac{4s^3 - s}{3}$$

Multiplications. The total number of multiplications in the new approach is then given by lists the ratio between the total number of multiplications required in the new and the classical approach for various values of $m$ and $n$. In all cases the new approach requires more multiplications than the classical approach. The minimum ratio is approximately 2.5 and obtained for the extreme (theoretical) case where $m = 1$ and $n = 106$. This ratio further increases primarily as a function of $m$. Indeed, increasing $m$ with a factor 10 from 102 to 103 (with $n = 105$ in both cases), increases the ratio with approximately a factor 10 as shown by columns 3 and 4. On the other hand, keeping $m$ constant and changing $n$ has a much smaller effect on the ratio as demonstrated by columns 4 and 5. In conclusion, for full $K_n$ and $M_n$, it is expected that the required computing time is much larger in the new approach when a significant set of masters is considered.

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS 7-13**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10^6$</td>
</tr>
<tr>
<td>10</td>
<td>$10^5$</td>
</tr>
<tr>
<td>$10^2$</td>
<td>$10^5$</td>
</tr>
<tr>
<td>$10^3$</td>
<td>$10^5$</td>
</tr>
<tr>
<td>$10^3$</td>
<td>$10^4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SSM</th>
<th>SEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>20.5</td>
</tr>
<tr>
<td>200.3</td>
<td>1978</td>
</tr>
<tr>
<td>1782</td>
<td></td>
</tr>
</tbody>
</table>

Ratio between number of multiplications required in the classical vs new approach. For full $K_n$ and $M_n$, the memory consumption, considered here as the number of nonzero elements, is comparable because both methods start with a system of dimension $n$ which is reduced by one after each iteration step ($K_{mm}$, $M_{mm}$ in the classical approach versus $K_{ss}$, $M_{ss}$ in the new approach).

**Computational efficiency: sparse $K_n$, $M_n$**

In the above, $K_n$ and $M_n$ were considered to be full while in practice they are sparse. Consequently, the number of multiplications and the storage requirements are drastically reduced. It seems reasonable to expect that the sparsity of $K_n$ and $M_n$ will not change the above derived conclusions regarding computing time and memory, i.e. the computing time required in the new approach still exceeds by far that of the classical approach while the storage needs are equivalent for both methods. For elementary FEM models (e.g. tridiagonal systems) this is indeed correct. However, for more complex FEM models, every step of the classical approach creates new nonzero elements in the initially sparse mass and stiffness matrices (so called fill-in), driving this procedure computationally expensive as shown hereafter. The classical approach on a component level:

$$K_m(i,j) = K_{mm}(i,j) - \frac{K_{ms}(i)K_{sm}(j)}{K_{ss}} \quad i, j = 1, \ldots, m$$  \hspace{1cm} (7.45)

If the coupling between elements $i$ and $j$ is zero prior to a reduction step ($K_{mm}(i, j) = 0$), Eq. (7.45) shows that a new nonzero element is created after the reduction step ($K_m(i,j) \neq 0$) when both
master coordinates i and j are coupled with the slave coordinate \((K_{ms}(i) \neq 0, K_{sm}(j) \neq 0)\). The same conclusion holds for \(M_{m(i,j)}\) derived from

\[
M_{m}(i, j) = M_{mm}(i, j) + M_{ss} \frac{K_{ms}(i)K_{sm}(j)}{K_{ss}} \left[ \frac{M_{ms}(i)K_{sm}(j) + K_{ms}(i)M_{sm}(j)}{K_{ss}} \right]
\]

(7.46)

In the simplest case, i.e. \(Km\) tri diagonal and \(Mm\) diagonal \((m = n\) initially), each reduction step creates two extra nonzero elements in \(Km\) and \(Mm\). As every reduction step reduces the size of \(Km\) and \(Mm\) with one,

**COMPUTATIONALLY EFFICIENT SELECTION OF MASTER DOFS 7-15**

The fill-in caused by the step by step Guyan reduction procedure is independent of the ordering of the DOFs, different re-ordering techniques exist to minimize the fill-in caused by factorization. Despite these advances, there is a tendency to prefer iterative methods when dealing with large mechanical structures. One of the most popular iterative techniques is the conjugant gradient (CG) method generally combined with a preconditioner to speed up the convergence. A detailed comparison between iterative and direct solvers is given in where it is shown that iterative solvers can provide advantages in both storage and computation time especially when 8-node hexahedral elements are considered. It is not possible to derive general conclusions regarding the computational efficiency of solving Eq. (7.47) because this highly depends on the solver and the sparsity pattern of \(Kn\) and \(Mn\). To give some idea, we apply a direct solver on the plate given. The number of nonzero elements after factorization is only 6.6 times higher than initially and can be further reduced to 3.3 by reordering. Compared to the number of nonzero elements after 300 iterations in the step by step Guyan reduction procedure, this is approximately a factor 10 lower. Besides this decreased storage need, the new approach also results in a factor 10 lower in computing time which can be even further reduced by applying iterative solvers.

**ACCURACY**

Applying Guyan reduction in each iteration step ensures that the effect of a condensed slave is redistributed over the remaining degrees of freedom. This is a crucial feature of the classical selection procedure which significantly increases the accuracy compared to the case where the slave DOFs are all selected at once. This redistribution is also achieved in the new approach by redefining \(Kss\) and \(Mss\) in each step, thereby taking into account the previously determined master DOFs. Furthermore, both methods are based on Eq. (7.26), which is derived from the Guyan reduction procedure itself.

**CONCLUSION**

The classical algorithm for Guyan reduction developed by leads to very high fill-in when complex FEM models (e.g. with 8-node solid elements) are considered. As a result, the computing time and required storage needs are substantially increased. To overcome this problem, a new method is proposed where the master DOFs are sequentially selected instead of the slave DOFs drastically reducing in this way the required number of iterations. It is shown that the classical as well as the new selection procedure are based on the same measure \(|K−1 sssMss|\) which denotes the relative error introduced by selecting a certain DOF as a slave. Consequently, a comparable accuracy of the reduced models is obtained.
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