International Journal of Mechanical Engineering and Technology (IJMET)
Volume 9, Issue 3, March 2018, pp. 272–282 Article ID: IJMET_09_03_028
Available online at http://www.iaeme.com/IJMET/issues.asp?JType=IJMET&VType=9&IType=3
ISSN Print: 0976-6340 and ISSN Online: 0976-6359

GENERALIZED (α, β)-RATIONAL CONTRACTIONS IN ORDERED S_b-METRIC SPACES WITH APPLICATIONS

B. Srinuvasa Rao, G.N.V.Kishore and Y. Hari Krishna
Department of Mathematics, KoneruLakshmaiah Education Foundation,
Vaddeswaram, Guntur, Andhra Pradesh, India

Muhammad Sarwar
Department of Mathematics, University of Malakand, ChakadaraDir(L),
Khyber Pakhtunkhwa, Pakistan

ABSTRACT

In this paper, we define generalized (α, β)-rational contraction based on this, we have to prove some fixed point theorems, applications related to integral equations and Homotopy theory. Also we gave an example which supported our main results.

Keyword: Fixed point, Generalized (α, β)-rational contraction, Homotopy theory, S_b -completeness, S_b -metric spaces, 2000 Mathematics Subject Classification: Primary 47H10; secondary 54H25.

Cite this Article: B. Srinuvasa Rao, G.N.V.Kishore, Y. Hari Krishna and Muhammad Sarwar, Generalized (α,β)-Rational Contractions in Ordered S_b-Metric Spaces with Applications, International Journal of Mechanical Engineering and Technology 9(3), 2018. pp. 272–282.
http://www.iaeme.com/IJMET/issues.asp?JType=IJMET&VType=9&IType=3

1 INTRODUCTION

In 1922, S. Banch [1] formulated the contraction known is Banach contraction principle. It is most fundamental tool in nonlinear analysis and some results related with generalization of various type of metric spaces (see [1]-[16]).

In 1989, Bakhtin formulated the b-metric spaces [2], later several researchers work on this space and obtained so many results on this space (see [7] - [10]).

In generalized contractions one is (α, β)-weak contraction. Using this contraction, researchers proved results (for detail see [17] - [23]). Currently the study of (α, β)-contractions gain the attractions of many researchers. In this regards many fixed point results and their applications are studied (see, [24] - [30] and the reference cited therein).

Mustafa et. al. defined the notion of G-metric space [3]. Sedghi et. al. gave the concept of an S-metric space [4]. Aghajan et. al. presented a new type of metric is called G_b-metric [5]. Recently, Sedghi et. al. [6] defined S_b –metric space by using the S-metric space [4].

The aim of present article is to prove applications to integral equations and Homotopy theory via generalized (α, β) –rational contraction, we can also gave related fixed point results and example.

Firstly, recall some definitions, lemmas and examples.

2 PRELIMINARIES

Definition 2.1: ([6]) Let S_b: X^3 →[0, 1) be a mapping defined on a non-empty set X and b≥ 1 be a given real number. Suppose that the mapping S_b satisfies the following properties:

(S_b1) 0<S_b(x, y, z) for all x, y, z ∈ X with x≠y $\neq z \neq x$

(S_b2) S_b(x, y, z)=0 ⇔ x= y = z

(S_b3) 0<S_b(x, y, z)≤ b (S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)) for all x, y, z, a∈ X. Then the function S_b is called a S_b-metric on X and the pair (X, S_b) is called a S_b-metric space.

Remark 2.2: ([6]) It must be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. In fact, each S-metric space is a S_b-metric space whenever b=1.

Following example shows that a S_b-metric space on X need not be a S-metric spaces

Example 2.3: ([6]) Let (X, S) be S-metric space and S_*(x, y, z) = $S(x,y,z)^p$ where p>1 is a real number. Note that (X, S_*) is not necessarily S-metric space but S_* is a S_b-metric with b=$2^{2(p-1)}$.

Definition 2.4: ([6]) Let (X, S_b) be a S_b-metric space. Then, we define the open ball$B_{S_b}(x,r)$ and closed ball B_{S_b}[x, r] with centre x∈ X and radius r> 0 as following respectively:

$B_{S_b}(x,r)$= {y∈X :S_b(y, y, x) <r} and B_{S_b}[x, r] = {y∈X :S_b(y, y, x)≤r}

Lemma 2.5: ([6]) In a S_b-metric space, we have S_b(u, u, w)≤2bS_b(u, u, v) +$b^2 S_b$(v, v, w).

Definition 2.6: ([6]) If (X, S_b) be a S_b-metric space. A sequence { x_n} in X is said to be

- S_b-Cauchy sequence if, for each $\epsilon > 0$, there is an integer $n_0 \in Z^+$ such that
- S_b(x_n , x_n, x_m)< ϵ for each n, m ≥ n_0
- S_b-convergent to a point x∈ X if, for each$\epsilon > 0$, there is an integer $n_0 \in Z^+$ such that S_b(x_n , x_n, x)< ϵ or S_b(x, x, x_n)< ϵ for all n≥ n_0 and denoted by$\lim_{n\to\infty} x_n$= x.

Definition 2.7: ([6]) A S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma 2.8: ([6]) If (X, S_b) be a S_b-metric space with b≥ 1 and suppose that {x_n} is a S_b-convergent to x, then we have

- $\frac{1}{2b}S_b$(y, y, x)≤ $\lim_{n\to\infty} inf S_b$(y, y, x_n) ≤ $\lim_{n\to\infty} sup S_b$(y, y, x_n) ≤ $2bS_b$(y, y, x) and
- $(ii)\frac{1}{b^2}S_b$(x, x, y)≤ $\lim_{n\to\infty} inf S_b(x_n, x_n$,y) ≤ $\lim_{n\to\infty} sup S_b(x_n, x_n$,y) ≤ $b^2 S_b$(x, x, y) for all y∈ X. Specifically, if x=y then $\lim_{n\to\infty} S_b(x_n, x_n$,y)=0

3. RESULTS AND DISCUSSIONS

Definition 3.1: Let (X, S_b) be a S_b-metric space, E: X→ X be say that to satisfy generalized (α,β)-rational contraction if there exists continuous maps α,β: [0, ∞) → [0, ∞) such

3.1.1. $\frac{1}{4b^3}\min\{S_b(\text{x},\text{x},\text{Ex}),S_b(\text{y},\text{y},\text{Ey})\} \le S_b(\text{x, x, y}) \Rightarrow$

$\alpha\big(4b^5S_b(\text{Ex},\text{Ex},\text{Ey})\big) \le \alpha\,({N_E}^i(\text{x, y}))\text{- }\beta\,({N_E}^i\,(\text{x, y}))$

For all x, y∈ X, x is comparable to y, i= 3 or 4 and

${N_E}^4(\text{x, y})=\max\{S_b(\text{x},\text{x},\text{y}),\ S_b(\text{x},\text{x},\text{Ex}),\ S_b(\text{y},\text{y},\text{Ey}),\ \frac{S_b(\text{x,x,Ey})S_b(\text{y,y,Ex})}{1+S_b(\text{x,x,y})+S_b(\text{Ex,Ex,Ey})}\}$

${N_E}^3(\text{x, y})=\max\{S_b(\text{x},\text{x},\text{y}),\ \frac{S_b(\text{x,x,Ex})S_b(\text{y,y,Ey})}{1+S_b(\text{x,x,y})+S_b(\text{Ex,Ex,Ey})},\ \frac{S_b(\text{x,x,Ey})S_b(\text{y,y,Ex})}{1+S_b(\text{x,x,y})+S_b(\text{Ex,Ex,Ey})}\}$

3.1.2. α(t) and β(t) vanish at t=0

3.1.3. β(t)> 0 for t>0.

Theorem 3.2: Let (X,S_b, ⪯) be complete ordered S_b-metric space, E: X → X satisfies generalized (α,β)-contraction with i= 4 and E is continuous or X is regular. If there exists x_0 ∈X with x_0⪯Ex_0. Then E has unique fixed point in X.

Proof: Since E is self- map, then there exists a sequence $\{x_n\}$ in X such that x_{n+1} = E x_n, n=0,1,2,3,……..

Case (i): If x_n = E x_n=x_{n+1}, then clearly proof is over.

Case (ii): Assume x_n ≠E x_n ∀ n.

Since x_0⪯Ex_0 = $x_1$and by the definition of E, we have

$x_0\preceq \text{E}x_0\preceq E^2x_0\preceq E^3x_0 \preceq\ldots\ldots\ldots\preceq E^nx_0\preceq E^{n+1}x_0\preceq\ldots\ldots\ldots$

Since $\frac{1}{4b^3}\min\{S_b(x_0,x_0,\text{E}x_0),S_b(x_1,x_1,\text{E}x_1)\} \le S_b(x_0,\,x_0,\,x_1)$

Now $\alpha\big(4b^5S_b(\text{E}x_0,\text{E}x_0,E^2x_0)\big) = \alpha\big(4b^5S_b(\text{E}x_0,\text{E}x_0,\text{E}x_1)\big)$

$\le \alpha\,({N_E}^i(x_0,\,x_1))\text{- }\beta\,({N_E}^i\,(x_0,\,x_1))$

Where

$${N_E}^4(x_0,\,x_1)=\max\left\{\begin{matrix}S_b(x_0,x_0,x_1),S_b(x_0,x_0,\text{E}x_0),S_b(x_1,x_1,\text{E}x_1)\\ \frac{S_b(x_0,x_0,\text{E}x_1)S_b(x_1,x_1,\text{E}x_0)}{1+S_b(x_0,x_0,\text{E}x_0)+S_b(\text{E}x_0,\text{E}x_0,\text{E}x_1)}\end{matrix}\right\}$$

$$=\max\left\{\begin{matrix}S_b(x_0,x_0,Ex_0),S_b(x_0,x_0,\text{E}x_0),S_b(Ex_0,Ex_0,E^2x_0)\\ \frac{S_b(x_0,x_0,E^2x_0)S_b(Ex_0,\text{E}x_0,\text{E}x_0)}{1+S_b(x_0,x_0,\text{E}x_0)+S_b(\text{E}x_0,\text{E}x_0,E^2x_0)}\end{matrix}\right\}$$

$$=\max\left\{S_b(x_0,x_0,Ex_0),S_b(Ex_0,Ex_0,E^2x_0)\right\}$$

Thus

$$\alpha\big(4b^5S_b(\text{E}x_0,\text{E}x_0,E^2x_0)\big) \le \alpha\left(\max\left\{S_b(x_0,x_0,Ex_0),S_b(Ex_0,Ex_0,E^2x_0)\right\}\right)$$
$$-\beta\left(\max\left\{S_b(x_0,x_0,Ex_0),S_b(Ex_0,Ex_0,E^2x_0)\right\}\right)$$

Also since $\frac{1}{4b^3}\min\{S_b(x_1,x_1,\text{E}x_1),S_b(x_2,x_2,\text{E}x_2)\} \le S_b(x_1,\,x_1,\,x_2)$

So that we have

$$\alpha\big(4b^5S_b(E^2x_0,E^2x_0,E^3x_0)\big) \le \alpha\left(\max\left\{S_b(Ex_0,Ex_0,E^2x_0),S_b(E^2x_0,E^2x_0,E^3x_0)\right\}\right)$$

$$-\beta\left(\max\left\{S_b(Ex_0, Ex_0, E^2x_0), S_b(E^2x_0, E^2x_0, E^3x_0)\right\}\right)$$

Continuing this way we can conclude that

$$\alpha\left(4b^5 S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)\right)$$
$$\le \alpha\left(\max\left\{S_b(E^n x_0, E^n x_0, E^{n+1}x_0), S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)\right\}\right)$$
$$-\beta\left(\max\{S_b(E^n x_0, E^n x_0, E^{n+1}x_0), S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)\}\right)$$

If $S_b(E^n x_0, E^n x_0, E^{n+1}x_0) < S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)$, we get contradiction.

Hence, $S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0) \le S_b(E^n x_0, E^n x_0, E^{n+1}x_0)$.

Thus $\{S_b(E^n x_0, E^n x_0, E^{n+1}x_0)\}$is non-increasing and must converges to a real number $\eta \ge 0$ (say). Also

$$\alpha\left(4b^5 S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)\right) \le \alpha\left(S_b(E^n x_0, E^n x_0, E^{n+1}x_0)\right)$$
$$-\beta\left(S_b(E^n x_0, E^n x_0, E^{n+1}x_0)\right)$$

Letting $n \to \infty$, we have $\alpha(4b^5\eta) \le \alpha(\eta) - \beta(\eta)$. It is clear that $\boldsymbol{\eta}=0$. That is $\lim_{n\to\infty} S_b(E^n x_0, E^n x_0, E^{n+1}x_0)=0$.

Now we prove that $\{E^n x_0\}$ is Cauchy sequence in (X, S_b). On contrary we suppose that $\{E^n x_0\}$ is not Cauchy. Then there exists $\epsilon > 0$ and monotonically increasing sequence of natural numbers natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$

$$S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) \ge \epsilon \qquad \textbf{(3.2.1)}$$

$$and\ S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) < \epsilon \qquad \textbf{(3.2.2)}$$

We claim that

$$\frac{1}{4b^3}\min\left\{S_b\left(x_{m_k}, x_{m_k}, Ex_{m_k}\right), S_b\left(x_{n_k-1}, x_{n_k-1}, Ex_{n_k-1}\right)\right\} \le S_b\left(x_{m_k}, x_{m_k}, Ex_{n_k-1}\right)$$

On contrary suppose that

$$\frac{1}{4b^3}\min\left\{S_b\left(x_{m_k}, x_{m_k}, Ex_{m_k}\right), S_b\left(x_{n_k-1}, x_{n_k-1}, Ex_{n_k-1}\right)\right\}$$
$$> S_b\left(x_{m_k}, x_{m_k}, Ex_{n_k-1}\right) \qquad \textbf{(3.2.3)}$$

Now consider

$$\epsilon \le S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0)$$
$$\le 2b\, S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) + b^2 S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0)$$
$$\le \frac{1}{2b^2}\min\left\{S_b(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0), S_b\left(x_{n_k-1}, x_{n_k-1}, Ex_{n_k}\right)\right\}$$
$$+ b^2 S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0)$$

Letting $k \to \infty$, it follows$\epsilon \le 0$. It is contradiction. Hence our claim (3.2.3)

From (3.2.1) and (3.2.2), we have

$$\epsilon \le S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0)$$
$$\le 2b\, S_b(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0) + b^2 S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)$$

Letting $k \to \infty$, we have

$$\phi(4b^3\epsilon) \le \lim_{k\to\infty} \phi(4b^5 S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0))$$
$$\le \lim_{k\to\infty} \alpha\left(4b^5 S_b\left(Ex_{m_k}, Ex_{m_k}, Ex_{n_k-1}\right)\right)$$
$$\le \lim_{k\to\infty,} \alpha\left(N^4{}_E\left(x_{m_k}, x_{n_k-1}\right)\right) - \lim_{k\to\infty,} \beta\left(N^4{}_E\left(x_{m_k}, x_{n_k-1}\right)\right) \qquad \textbf{(3.2.4)}$$

Where, $\lim_{k\to\infty} N^4{}_E\left(x_{m_k}, x_{n_k-1}\right)$

$$= \lim_{k\to\infty} max \left\{ \begin{matrix} S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right), S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0\right), \\ S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0) \\ \frac{S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0)}{1+S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right)+S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)} \end{matrix} \right\}$$

But

$$\lim_{k\to\infty} \left\{ \frac{S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0)}{1 + S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) + S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)} \right\}$$

$$= \lim_{k\to\infty} \left\{ \left(\frac{[2bS_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) + b^2 S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0)] \frac{2bS_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k}x_0)+b^2 S_b(E^{m_k}x_0, E^{m_k}x_0, E^{m_k}x_0)}{}}{1+S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0)+S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)} \right) \right\}$$

$$\leq 4b^3\epsilon$$

Now from (3.2.4), we have $\alpha(4b^3\epsilon) \leq \alpha(4b^3\epsilon)$- $\lim_{k\to\infty,} \beta\left(N^4{}_E(x_{m_k}, x_{n_k-1})\right) < \alpha(4b^3\epsilon)$

It is contradiction. Hence $\{E^n x_0\}$ is a Cauchy sequence in (X, S_b). Because of completeness of (X, S_b), there is an $\upsilon \in X$ with $\{E^n x_0\} \to \upsilon \in$(X, S_b). That is $\lim_{n\to\infty} E^n x_0 = \upsilon = \lim_{n\to\infty} E^{n+1} x_0$.

First we claim that for each n≥1, at least one of the following assertion is holds.

$\frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) \leq S_b(\upsilon, \upsilon, x_n)$ or $\frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) \leq S_b(\upsilon, \upsilon, x_{n-1})$

On contrary suppose that

$\frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) > S_b(\upsilon, \upsilon, x_n)$ and $\frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) > S_b(\upsilon, \upsilon, x_{n-1})$

Now consider $S_b(x_{n-1}, x_{n-1}, x_n) \leq 2b\, S_b(x_{n-1}, x_{n-1}, \upsilon) + b^2 S_b(\upsilon, \upsilon, x_n)$

$$< 2b^2 S_b(\upsilon, \upsilon, x_{n-1}) + b^2 \frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n)$$

$$< 2b^2 \frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) + \frac{1}{4b} S_b(x_{n+1}, x_{n+1}, x_n)$$

$$< \frac{1}{2b} b\, S_b(x_{n-1}, x_{n-1}, x_n) + \frac{1}{4b} b S_b(x_n, x_n, x_{n+1})$$

$$< \frac{1}{2} S_b(x_{n-1}, x_{n-1}, x_n) + \frac{1}{4b^3} S_b(x_{n-1}, x_{n-1}, x_n)$$

$$= \frac{2b^3+1}{4b^3} S_b(x_{n-1}, x_{n-1}, x_n) < \frac{3}{4} S_b(x_{n-1}, x_{n-1}, x_n).$$

It is contradiction. Hence our claim is holds. Since E$x_n \to \upsilon$ and (X, S_b) is regular, it follows that x_n is comparable to υ.

Suppose E υ ≠ υ, from (3.1.1) and the definition of α, Lemma (2.8), we have

$\alpha\left(2b^4 S_b(\mathrm{E}\,\upsilon, \mathrm{E}\,\upsilon, \upsilon)\right) \leq \lim_{n\to\infty} inf \alpha\left(2b^4 S_b(\mathrm{E}\,\upsilon, \mathrm{E}\,\upsilon, E^{n+1}x_0)\right)$

$\leq \lim_{n\to\infty,} \alpha(N^4{}_E(\upsilon, x_n))$- $\lim_{n\to\infty,} \beta(N^4{}_E(\upsilon, x_n))$ **(3.2.5)**

Here,

$$\lim_{n\to\infty} N_E{}^4(\upsilon, x_{n_k}) = \lim_{n\to\infty} max \left\{ \begin{matrix} S_b\left(\upsilon, \upsilon, x_{n_k}\right), S_b(\upsilon, \upsilon, \mathrm{E}\upsilon,), S_b(x_{n_k}, x_{n_k}, \mathrm{E}x_{n_k}) \\ \frac{S_b\left(\upsilon, \upsilon, \mathrm{E}x_{n_k}\right) S_b(x_{n_k}, x_{n_k}, \mathrm{E}\upsilon,)}{1+S_b\left(\upsilon, \upsilon, x_{n_k}\right)+S_b\left(\mathrm{E}\upsilon, \mathrm{E}\upsilon, Ex_{n_k}\right)} \end{matrix} \right\}$$

$$= S_b(\upsilon, \upsilon, \mathrm{E}\upsilon,)$$

Hence from (3.2.5), we have

$\alpha\left(2b^4 S_b(\mathrm{E}\,\upsilon, \mathrm{E}\,\upsilon, \upsilon)\right) \leq \alpha\left(S_b(\upsilon, \upsilon, \mathrm{E}\,\upsilon)\right)$- $\beta\left(S_b(\upsilon, \upsilon, \mathrm{E}\upsilon)\right)$

$$\leq \alpha\left(S_b(\upsilon, \upsilon, \mathrm{E}\,\upsilon)\right)$$

Clearly υ is fixed point of E.

Assume that υ^* is also fixed point of E such that $\upsilon \neq \upsilon^*$.

Since $\frac{1}{4b^3}$ min $\{S_b(\upsilon,\upsilon,E\upsilon), S_b(\upsilon^*,\upsilon^*,E\upsilon^*)\} \leq S_b(\upsilon,\upsilon,\upsilon^*)$

Consider $\alpha\big(4b^5\, S_b(\upsilon,\upsilon,\upsilon^*)\big) \leq \alpha(N^4{}_E(\upsilon,\upsilon^*))$- $\beta(N^4{}_E(\upsilon,\upsilon^*))$

$= \alpha\big(S_b(\upsilon,\upsilon,\upsilon^*)\big)$- $\beta\big(S_b(\upsilon,\upsilon,\upsilon^*)\big)$

$\leq \alpha\big(S_b(\upsilon,\upsilon,\upsilon^*)\big)$

Clearly υ is unique fixed point of E in (X, S_b).

Example 3.3: LetS_b: X×X×X→ R^+ be a mapping defined as

S_b(u, v, w)=$(|v+w-2u|+|v-w|)^2$ where X= [0, 1] and ⪯ by u⪯w⇔ u $\leq w$. So clearly

(X, S_b, ⪯) is complete ordered S_b-metric space with b=4. Define E: X→X by E(u)= $\frac{u}{4^3}$ also define α,β: $R^+ \to R^+$ by α (t)=t and β(t)=$\frac{(b-1)t}{b}$, then

$\alpha\big(4b^5\, S_b(\text{Eu},\text{Eu},Ev)\big)$ =$\boldsymbol{4b^5}(|Eu+Ev-2Eu|+|Eu-Ev|)^2$+r

=$\boldsymbol{4b^5}\left(2\left|\frac{u}{4^3}-\frac{v}{4^3}\right|\right)^2$+r $\leq \frac{1}{b}\, S_b(\text{u},\text{u},v) \leq \alpha\big(N^4{}_E(\text{u},v)\big)$- $\beta\big(N^4{}_E(\text{u},v)\big)$

Therefore, all the conditions of Theorem 3.2 satisfied and 0 is unique fixed point of E.

Theorem 3.4: Let (X,S_b, ⪯) be a complete ordered S_b-metric space, E : X → X satisfies generalized (α,β)-contraction with i= 3 and E is continuous or X is regular. If there exists x_0 ∈X with x_0⪯Ex_0. Then E has unique fixed point in X.

Proof: If we replace $N^3{}_E(\text{x},y)$ in place of $N^4{}_E(\text{x},y)$, the rest of proof follows from Theorem 3.2.

Theorem 3.5: Let (X,S_b, ⪯) be a complete ordered S_b-metric space and let E: X → X be satisfies.

(3.5.1) $\frac{1}{4b^3}$ min $\{S_b(\text{x},\text{x},\text{Ex}), S_b(\text{y},\text{y},\text{Ey})\} \leq S_b(\text{x, x, y}) \Rightarrow$

$4b^5\, S_b(\text{Ex},\text{E x},\text{Ey}) \leq N^i{}_E(\text{x},y)$- $\beta\big(N^i{}_E(\text{x},y)\big)$

Where β: $[0,\infty) \to [0,\infty)$ continuous with β(t)>0 for t>0 and i= 3, 4 and E is continuous or X is regular. If there exists x_0 ∈X with x_0⪯Ex_0. Then E has unique fixed point in X.

Theorem 3.6: Let (X,S_b, ⪯) be a complete ordered S_b-metric space and let E: X → X be satisfies.

(3.6.1) $\frac{1}{4b^3}$ min $\{S_b(\text{x},\text{x},\text{Ex}), S_b(\text{y},\text{y},\text{Ey})\} \leq S_b(\text{x, x, y}) \Rightarrow$

$S_b(\text{Ex},\text{E x},\text{Ey}) \leq \lambda N^i{}_E(\text{x},y)$, where λ∈[0, $\frac{1}{4b^5}$) and i= 3, 4 and E is continuous or X is regular. If there exists x_0 ∈X with x_0⪯Ex_0. Then E has unique fixed point in X.

4. APPLICATIONS TO INTEGRAL EQUATIONS

Theorem 4.1: Consider the I.V.P $\boldsymbol{u'}$(x)= P (x, u(x)); x∈I=[0,1], u(0)= $\boldsymbol{u_0}$ (4.1.1)

Where P: I×[$\frac{u_0}{4}$, ∞) →[$\frac{u_0}{4}$, ∞) and u_0 ∈R. Then (4.1.1) has a unique solution.

Proof: The integral equation of I.V.P (4.1.1) is u(x)= u_0 + $3b^3\int_0^x P\big(t,u(t)\big)dt.$

Let X = C (I, $[\frac{u_0}{4},\infty)$) and $S_b(u,v,w) = (|v+w-2u| + |v-w|)^2$ for $u,v,w \in$ X. Then (X,$\boldsymbol{S_b}$) is a complete $\boldsymbol{S_b}$-metric space, also define α,β: [0, ∞) → $[0,\infty)$by α(x)= x and β(x)=$\frac{(9b-4)x}{9b}$.

Define E: X → X by E ($\boldsymbol{u}$) (x) = $\frac{\boldsymbol{u_0}}{\boldsymbol{3b^3}} + \int_0^x P\big(t,u(t)\big)dt.$ **(4.1.2)**

Now,

$$(4b^5 S_b(\mathrm{E}\boldsymbol{u}(\mathrm{x}), \mathrm{E}\boldsymbol{u}(\mathrm{x}), \mathrm{E}\boldsymbol{v}(\mathrm{x}))) = 4b^5\{|\ \mathbf{E}u(\mathbf{x}) + \mathbf{E}v(\mathbf{x}) - 2\,\mathbf{E}u(\mathbf{x})|\ +\ |\mathbf{E}u(\mathbf{x}) - \mathbf{E}v(\mathbf{x})|\}^2$$
$$= 16b^5|\mathbf{E}u(\mathbf{x}) - \mathbf{E}v(\mathbf{x})|^2$$
$$= \frac{16b^5}{9b^6}\left|\boldsymbol{u_0} + 3b^3\int_0^x P\big(t,u(t)\big)dt - \boldsymbol{v} - 3b^3\int_0^x P\big(t,v(t)\big)dt\right|^2$$
$$= \frac{16}{9b}|u(x) - v(x)|^2$$
$$\leq \frac{4}{9b}\mathrm{S}(u,u,v) \leq \alpha(N^4{}_E(\mathrm{u},v))\beta(N^4{}_E(\mathrm{u},v))$$

It follows from Theorem 3.2, E has a unique fixed point in X.

5 APPLICATIONS TO HOMOTOPY

Theorem 5.1: Let (X,$\boldsymbol{S_b}$) be complete $\boldsymbol{S_b - metric}$ space, U and $\overline{\boldsymbol{U}}$ be an open and closed subset of X such that U ⊆$\overline{\boldsymbol{U}}$. Assume that H_b: $\overline{\boldsymbol{U}}$ × [0,1] → X be an operator satisfying the following conditions.

(**5.1.1**) u ≠, H_b(u, κ) for each u ∈ ∂U and $\kappa \in$ [0,1] ($Here$ ∂U is boundary of U in X),

(**5.1.2**) $\frac{1}{4b^3}$ min $\{S_b(\mathrm{u},\mathrm{u},H_b(\mathrm{u},\kappa)),S_b(\mathrm{v},\mathrm{v},H_b(\mathrm{v},\kappa))\} \leq S_b(\mathrm{u},\mathrm{u},\mathrm{v}) \Rightarrow$

$\alpha(4b^5(\boldsymbol{S_b}(H_b(\mathrm{u},\kappa),H_b(\mathrm{u},\kappa),H_b(\mathrm{v},\kappa)))) \leq \alpha\ (\boldsymbol{S_b}(\mathbf{u},\mathbf{u},\mathbf{v}))\ \beta\ (\boldsymbol{S_b}(\mathrm{u},\mathrm{u},\mathrm{v}))$

For all u, v ∈ $\overline{\boldsymbol{U}}$and κ ∈ [0,1], where α and β are defined in Theorem (3.2),

(5.1.3) $\exists M_b \geq 0 \ni \boldsymbol{S_b}(H_b(\mathrm{u},\kappa),H_b(\mathrm{u},\kappa),H_b(\mathrm{u},\zeta)) \leq M_b|\kappa - \zeta|$. For every u ∈ $\overline{\boldsymbol{U}}$ and $\kappa,\zeta \in$ [0,1]. Then H_b(., 0) has a fixed point ⇔ H_b(.,1) has a fixed point.

Proof Let the set B = { $\kappa \in$ [0, 1]: u = H_b(u, κ) for some u ∈ U}.

Since H_b(., 0) has a fixed point in U, so 0 ∈ B.

Now, prove B is closed as well as open in [0, 1] and hence by the connectedness B = [0,1]. Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq$ B with $\kappa_n \to \kappa \in$ [0, 1] as n → ∞.

Now, $\kappa \in$ B must be shown. Since $\kappa_n \in$ B for n = 0, 1, 2, 3,…..there exists $u_n \in$ U with $u_n = H_b(u_n, \kappa_n)$.

Consider

$$\boldsymbol{S_b}(\boldsymbol{u_n},\boldsymbol{u_n},\boldsymbol{u_{n+1}}) = \boldsymbol{S_b}(H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n+1}}))$$
$$\leq 2\mathrm{b}\boldsymbol{S_b}(H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n}}))$$
$$+\ \boldsymbol{b^2}\boldsymbol{S_b}(H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n+1}}))$$
$$\leq 2\mathrm{b}\boldsymbol{S_b}(H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n}})) + \boldsymbol{b^2}\mathrm{M}\ |\kappa_{\boldsymbol{n}} - \kappa_{\boldsymbol{n+1}}|.$$

Letting n → ∞

$$\lim_{n\to\infty}\boldsymbol{S_b}(\boldsymbol{u_n}, \boldsymbol{u_n}, \boldsymbol{u_{n+1}}) \leq \lim_{n\to\infty} 2\mathrm{b}\boldsymbol{S_b}(H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n}})) + 0.$$

Since $\frac{1}{4b^3}$ min $\{S_b(\boldsymbol{u_n},\boldsymbol{u_n},H_b(\boldsymbol{u_n},\kappa)),S_b(\boldsymbol{u_{n+1}},\boldsymbol{u_{n+1}},H_b(\boldsymbol{u_{n+1}},\kappa))\} \leq S_b(\boldsymbol{u_n}, \boldsymbol{u_n}, \boldsymbol{u_{n+1}})$

Therefore, from (5.1.2), we have

$$\lim_{n\to\infty}\alpha(2b^4\boldsymbol{S_b}(\boldsymbol{u_n},\boldsymbol{u_n},\boldsymbol{u_{n+1}}))\leq \lim_{n\to\infty}\alpha(4b^5\boldsymbol{S_b}(H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}), H_b(\boldsymbol{u_n}, \kappa_{\boldsymbol{n}}, H_b(\boldsymbol{u_{n+1}}, \kappa_{\boldsymbol{n}})))$$

$$\leq \lim_{n\to\infty} (\alpha(\boldsymbol{S_b}(\boldsymbol{u_n}, \boldsymbol{u_n}, \boldsymbol{u_{n+1}})))-\beta(\boldsymbol{S_b}(\boldsymbol{u_n}, \boldsymbol{u_n}, \boldsymbol{u_{n+1}}))$$

It follows that $\lim_{n\to\infty} \boldsymbol{S_b}(\boldsymbol{u_n},\boldsymbol{u_n},\boldsymbol{u_{n+1}})=0$ **(5.1.4)**

Now, $\{\boldsymbol{u_n}\}$ is a $\boldsymbol{S_b}$-Cauchy sequence in (X, $\boldsymbol{S_b}$) is to be shown. On contrary assume that $\{\boldsymbol{u_n}\}$ is not a $\boldsymbol{S_b}$-Cauchy sequence.

There is an $\boldsymbol{\epsilon} > 0$ and monotone increasing sequence of natural numbers $\{\boldsymbol{m}_k\}$ and $\{\boldsymbol{n}_k\}$ such that $\boldsymbol{n}_k > \boldsymbol{m}_k$,

$$\boldsymbol{S_b}(\boldsymbol{u_{m_k}}, \boldsymbol{u_{m_k}}, \boldsymbol{u_{n_k}}) \geq \boldsymbol{\epsilon} \quad \textbf{(5.1.5)}$$

And

$$\boldsymbol{S_b}(\boldsymbol{u_{m_k}}, \boldsymbol{u_{m_k}}, \boldsymbol{u_{n_{k-1}}}) < \boldsymbol{\epsilon}. \quad \textbf{(5.1.6)}$$

Therefore from (5.1.5) and (5.1.6),

$$\epsilon \leq \boldsymbol{S_b}(\boldsymbol{u_{m_k}}, \boldsymbol{u_{m_k}}, \boldsymbol{u_{n_k}})$$
$$\leq \boldsymbol{2bS_b}(\boldsymbol{u_{m_k}}, \boldsymbol{u_{m_k}}, \boldsymbol{u_{m_k+1}}) + b^2\boldsymbol{S_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{u_{m_k+1}}, \boldsymbol{u_{n_k}})$$

Letting k→ ∞, we have

$$\frac{\epsilon}{b^2} \leq \lim_{n\to\infty} \boldsymbol{S_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{u_{m_k+1}}, \boldsymbol{u_{n_k}})$$

But $\lim_{n\to\infty} \alpha(\boldsymbol{S_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{u_{m_k+1}}, \boldsymbol{u_{n_k}}))$

$$\leq \lim_{n\to\infty} \boldsymbol{\alpha}\left(\boldsymbol{9b^4S_b}\left(\boldsymbol{H_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{\kappa_{m_k+1}}), \boldsymbol{H_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{\kappa_{m_k+1}}), \boldsymbol{H_b}(\boldsymbol{u_{n_k}}, \boldsymbol{\kappa_{n_k}})\right)\right)$$
$$\leq \boldsymbol{\lim_{n\to\infty}}\left(\boldsymbol{\alpha}(\boldsymbol{S_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{u_{m_k+1}}, \boldsymbol{u_{n_k}}))\right) - \boldsymbol{\beta}(\boldsymbol{S_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{u_{m_k+1}}, \boldsymbol{u_{n_k}})))$$

It follows that

$\lim_{n\to\infty} \boldsymbol{S_b}(\boldsymbol{u_{m_k+1}}, \boldsymbol{u_{m_k+1}}, \boldsymbol{u_{n_k}}) \leq 0$ and hence $\boldsymbol{\epsilon} \leq \boldsymbol{0}$. It is a contradiction.

Hence $\{u_n\}$ is a S_b- Cauchy sequence in (X, S_b). By completeness there exists $\eta \in$ U such that

$$\lim_{n\to\infty} u_n = \eta = \lim_{n\to\infty} u_{n+1} \quad \textbf{(5.1.7)}$$

Since $\frac{1}{4b^3}$ min $\{S_b(\eta, \eta, H_b(\eta, \kappa)), S_b(\boldsymbol{u_n}, \boldsymbol{u_n}, H_b(\boldsymbol{u_n}, \kappa))\} \leq S_b(\boldsymbol{\eta}, \eta, \boldsymbol{u_n})$

Now,

$$\alpha\left(\frac{1}{2b} S_b(H_b(\eta, \kappa), H_b(\eta, \kappa), \eta)\right) \leq \lim_{n\to\infty} inf\ \alpha\left(S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), H_b(u_n, \kappa)\right)\right)$$
$$\leq \lim_{n\to\infty} inf\ \alpha\left(4b^5 S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), H_b(u_n, \kappa)\right)\right)$$
$$\leq \lim_{n\to\infty} \inf(\alpha(S_b(\eta, \eta, u_n)) - \beta(S_b(\eta, \eta, u_n)))=0$$

It follows that $H_b(\eta, \kappa)=\boldsymbol{\eta}$.

Thus κ∈ B. Clearly B is closed in [0, 1]. Let $\kappa_0 \in$ B. Then there exists $u_0 \in$ U such that $u_0=H_b(u_0, \kappa_0)$. Since U is open, then there exists r>0 such that $B_{S_b}(u_0, r) \subseteq$ U.

Choose $\kappa \in (\kappa_0\text{-}\epsilon, \kappa_0+\epsilon)$ such that $|\kappa - \kappa_0| \leq \frac{1}{M^n} < \epsilon$.

Then for u∈ $\overline{B_p}(u_0, r)=\{u \in X / S_b(u, u, u_0) \leq r + b^2 S_b(u_0, u_0, u_0)\}$. Also

$\frac{1}{4b^3}$ min $\{S_b(\mathrm{u}, \mathrm{u}, H_b(\mathrm{u}, \kappa)), S_b(\boldsymbol{u_0}, \boldsymbol{u_0}, H_b(\boldsymbol{u_0}, \kappa))\} \leq S_b(\mathrm{u}, \mathrm{u}, \boldsymbol{u_0})$ implies

$S_b(H_b(u, \kappa), H_b(u, \kappa), u_0) = S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u_0, \kappa_0))$

$\leq 2\mathrm{b} S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \kappa_0))$

$$+b^2S_b(H_b(u,\kappa_0),H_b(u,\kappa_0),H_b(u_0,\kappa_0))$$
$$\leq 2bM|\kappa-\kappa_0| + b^2S_b(H_b(u,\kappa_0),H_b(u,\kappa_0),H_b(u_0,\kappa_0))$$
$$\leq \frac{2b}{M^{n-1}} + b^2S_b(H_b(u,\kappa_0),H_b(u,\kappa_0),H_b(u_0,\kappa_0))$$

Letting $n \to \infty$ and applying α on both sides, then

$$\alpha\big(S_b(H_b(u,\kappa),H_b(u,\kappa),x_0)\big) \leq \alpha\big(b^2S_b(H_b(u,\kappa_0),H_b(u,\kappa_0),H_b(u_0,\kappa_0))\big)$$
$$\leq \alpha\big(4b^5S_b(H_b(u,\kappa_0),H_b(u,\kappa_0),H_b(u_0,\kappa_0))\big)$$
$$\leq \alpha\big(S_b(u,u,u_0)\big) - \beta(S_b(u,u,u_0))$$
$$\leq \alpha\big(S_b(u,u,u_0)\big)$$

Therefore,

$S_b(H_b(u,\kappa),H_b(u,\kappa),u_0) \leq S_b(u,u,u_0) \leq r + b^2S_b(u_0,u_0,u_0)$.

Thus for each fixed $\kappa \in (\kappa_0\text{-}\epsilon,\ \kappa_0\text{+}\epsilon)$, $H_b(.\ ,\ \kappa)$:$\overline{B_p}(u_0,r) \to \overline{B_p}(u_0,r)$. Then all the conditions of Theorem (5.1) are satisfied. Thus, we conclude that $H_b(.\ ;\ \kappa)$ has a fixed point in $\overline{U}$. But this must be in U. Therefore, κ∈B for $\kappa \in (\kappa_0\text{-}\epsilon,\ \kappa_0\text{+}\epsilon)$. Hence $(\kappa_0\text{-}\epsilon,\ \kappa_0\text{+}\epsilon)$⊆B. Clearly B is open in [0, 1].

Similar process can be used to prove the converse.

6. CONCLUSIONS

This paper presents some fixed point results by using (α,β)- rational contractive conditions defined on ordered $\boldsymbol{S_b}$-metric spaces and prasents suitable examples that supports the main results. Also, applications to integral equations as well as Homotopy theory are provided. This study will help researchers to generalized different contractions in S_b – metric spaces with applications to integral equations as well as Homotopy theory. Thus, a new framework on S_b – metric spaces may be arrived at.

DECLARATION

- **Competing Interest:** The authors declare that they have no competing interest.
- **Funding:** No funding.
- **Author's contributions:** All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

ACKNOWLEDGEMENTS

The authors are very thankful to the reviewers and editors for their valuable comments, remarks and suggestions which improved the paper in good form.

REFERENCES

[1] S. Banach, Theorie des Operations lineaires, Manograic Mathematic Zne, Warsaw, Poland, 1932.

[2] I. A. Bakhtin, The contraction mapping principle in quasi metric spaces, Funct. Anal. Unianowsk Gos, ped. Inst., 30(1989), 26-37.

[3] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J Nonlinear convex Anal, 7(2), (2006), 289-297.

[4] S. Sedghi, N. Shobe, and A. Aliouche, A generalization of fixed point theorem in S-metric spaces, Mat. Vesnik, 64(2012), 258-266.

[5] A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, Filomat, 28(6) (2014), 1087-1101.

[6] S. Sedghi, Gholidahneh, T. Dosenovic, J. Esfahani, and S. Radenovic, Common fixed point of four maps in S_b-metric spaces. Journal of Linear and Topological Algebra, Vol.5 (2),(2016), 93-104.

[7] S. Czerwik, Contraction mapping in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis, 1(1993), 5-11.

[8] V. Lakshmikantham, Lj. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis. Theory, Methods and Applications, 70(12), (2009), 4341-4349.

[9] Z. Mustafa, J R. Roshan, V. Parvanesh, and Z. Kadelburg, Some common fixed point results in ordered partial b-metric spaces, Journal of Inequalities and applications 2013(1), 562.

[10] Z. Mustafa, J R. Roshan, V. Parvanesh, and Z. Kadelburg, Fixed point theorems for Weakly T-chatterjea and weakly T-kannan contractions in b-metric spaces, Journal of Inequaliturs and Applications 2014, 2014:46.

[11] Z. Mustafa, M. Jaradat, A. H.Ansari, B.Z.Popovi, and H. Jaradat, C-class functions with new approach on coincidence point results for generalized generalized (α, β)-weak contractions in ordered b-metric spaces. Springer plus, 5:802. DOI: 10.1186/s40064-016-2481-1.

[12] Z. Mustafa, M.M.M. Jaradata, and H. M. Jaradat, Some common fixed point results of graphs on b-metric spaces. Journal of Nonlinear Science and Applications, 9(2016). Issue: 6, 4838-4851.

[13] M.M.M. Jaradat, Z. Mustafa, A. H. Ansari, P.S Kumari, D. D. Djekic, and H. M. Jaradat, Some fixed point results for $F_{\alpha-\omega,\varphi}$- generalized cyclic contraction on metric- like space with applications to graphs and integral equations, Journal of Math-ematical Analysis, Volume 8, issue 1 (2017), pages 28-45.

[14] G.N.V. Kishore, K.P.R. Rao, and V. M.L. Hima Bindu, Umique common fixed point theorem in partial metric spaces by using (C): condition with rational expressions, Afr. Mat. (2016), DOI 10.10.1007/s13370-017-0484-x

[15] S. Sedghi, I. Altun, N. Shobe, and M. Salahshour, Some properties of S-metric space and fixed point results, Kyung pook Math. J., 54(2014), 113-122.

[16] S. Sedghi, N. Shobe, and T. Dosenovic, fixed point results in S- metric spaces, Nonlinear Functional Analysis and Applications, 20(1), (2015), 55-67.

[17] D. Doric, Common fixed point for generalized (α, β)-weak contractions, Appl. Math. Lett, 22(2009), 1896-1900.

[18] M. Abbas, and M. Doric, Common fixed point for generalized (α, β)-weak contractions. Math. Un of Nis. Serbia. 10(2010), 1-10.

[19] P. N. Dutta, B. S. Choudhury, A generalization of contraction principle in metric spaces. Fix. Point theo. Appl, 18(2008), Article ID 406368.

[20] P. Parvateesam, Murthy, Kenan Tas, Uma Devi Patel, Common fixed point theorems for generalized (β, α)-weak contraction. Jour. Ineq; appl., 2015, DOI: 0.1186/s13660-015-0647-y.

[21] K.P.R. Rao, I. Altun, K.R.K. Rao, N. Srinivasarao, A Common fixed point theorem for Four maps under (α, β)-contractive condition of integral type in ordered partial metric spaces, Math. Sci. Lett., 4 No. 1, 25-31(2015).

[22] S. Radenivic, Z. Kadelburg, Fixed point results in partially ordered metric spaces, Appl. Math. Comput., 60 (2010), 1776-1783.

[23] H. Nashine, B. Samet, Fixed point results for mappings satisfying (α,β)-weakly contractive condition in partially ordered metric spaces, Non. Linear. Anal., 74, 2201-2209, (2011).

[24] A. Asadollah, M. Abbas, J.R. Roshan, Common fixed point results for four mappings satisfy generalized contractive condition in partially order b-metric spaces, Math-slovaca, 64 (2014), No. 4, 941-960.

[25] H. Huang, S. Randenovic, J. Vujakovic, Coincidence point results in partially ordered b-metric spaces, Appl. Math. Comput., (2015), 2015:63.

[26] J.R.Roshan, V. Parvaneh, I. Altun, Coincidence point results in partially ordered b-metric spaces, Appl. Math. And comput., 226(2014), 725-737.

[27] J.R.Roshan, V. Parvaneh, Z. Kadelburg, Fixed point theorem for weakly isotone increasing mapping in ordered b-metric spaces, J. Nonline. Sci. Appl., 7(2014), 229-245.

[28] J.Roshan, V. Parvanesh, S. Radenovic, and M. Rajovic, Coincidence point results for generalized (α,β)-weak contractions in ordered b-metric spaces. Fix. Point Theo. Appl, (2015), 2015:68.

[29] R. Allahyari, R. Arab, A.S. Haghighi, Coincidence point result of almost generalized contractive condition to partially order complete b-metric space and Application to quadratic integral equation, Jour. Ineq. Appl., 2014, 2014:355.

[30] J. Roshan, V. Parvaneh, N. Shobkolaei, S. Sedghi, W. Shatanawi, Common fixed point of almost generalized $(\beta,\alpha)_s$- contractive mappings in ordered b-metric spaces, Fix. Point Theo. Appl., 2013, Article ID 159 (2013)