ON VORTEX SHEET INTENSITY COMPUTATION FOR AIRFOILS WITH ANGLE POINT IN VORTEX METHODS

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ABSTRACT

Two-dimensional problem of numerical flow simulation around an airfoil is considered. The flow is assumed to be incompressible, so meshless lagrangian vortex method can be used to simulate the flow. The aim of this research is to investigate the accuracy of two different approaches to boundary conditions satisfaction, namely N-scheme and T-scheme, in case of flow simulation around airfoils with angle points. The asymptotic behavior of the solution, which is singular in the neighborhood of the angle point, is known. Different algorithms of the airfoil discretization into panels are implemented, and the accuracy of the considered numerical schemes is estimated. The obtained results show that the T-scheme provides more accurate solution in all considered cases in comparison to the N-scheme, especially at non-uniform airfoil discretization.

Key words: Vortex Method, Boundary Integral Equation, Angle Point, Non-Uniform Discretization.

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1. INTRODUCTION

Meshless purely Lagrangian vortex methods [1, 2, 3] can be efficient tool for numerical simulation in 2D problems of flow simulation around airfoils as well as in FSI-problems. Vortex methods are applicable in case of small subsonic velocities when compressibility influence can be neglected. They are especially efficient in external flows simulation around airfoil, when the flow domain is unbounded. Also, a significant advantage of lagrangian vortex methods is that the simulation of the flow around movable and/or deformable airfoils has nearly the same computational complexity as in case of immovable rigid airfoil.
Viscous incompressible flow is described by Navier–Stokes equations

\[ \nabla \cdot \vec{V} = 0, \quad \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\nabla p + \frac{\nu}{\rho} \Delta \vec{V} \]

Where \( \vec{V} \) is flow velocity; \( p \) is pressure; \( \rho \) is density; \( \nu \) is kinematic viscosity coefficient of the media.

The initial distribution of flow velocity is assumed to be known; decay condition is set at infinity (i.e., velocity and pressure tend to the corresponding given values for incident flow) and no-slip boundary condition is set on the airfoil surface.

\[ \vec{V}(\vec{r}) \to \vec{V}_\infty \text{ at } |\vec{r}| \to \infty; \quad \vec{V}\big|_{\vec{r} \in \Gamma} = \vec{V}_k. \]

Here \( \vec{V}_k \) is airfoil surface velocity at the corresponding points, which is assumed to be known; \( \vec{n} \) is outer normal unit vector.

Incompressible flow simulation by using vortex methods is based on the airfoil replacement with vortex and source sheets, placed on its surface line. However, some other approaches are also known, in particular, ‘virtual’ vorticity distribution can be set inside the airfoil in order to simulate its rotational motion [4]. The source sheet is assumed to be attached, its consideration is necessary in case of movable or deformable airfoil. The vortex sheet in general case consists of two parts – attached and free sheets; free vorticity is generated on the airfoil surface, then according to vorticity flux model [5] it becomes part of vorticity in flow domain and forms vortex wake near and behind the airfoil. The intensities of the attached vortex and source sheets are found from the airfoil surface line velocity and equal to its tangent and normal components, respectively. The unknown intensity of free vortex sheet can be found from no-slip boundary condition. There are several approaches to numerical schemes construction for vortex methods that differ in the way of boundary condition satisfaction [6, 7, 8].

Thus, flow simulation around the airfoil includes the solution of two main problems: free vortex sheet intensity computation and vorticity evolution simulation in the flow domain. The accuracy of vortex sheet intensity computation is the main factor, which determines the accuracy and the correctness of whole problem solution. The aim of this paper is to estimate the accuracy of the above mentioned ‘normal’ and ‘tangent’ approaches in case of flow simulation around airfoils with angle point.

2. BRIEF DESCRIPTION OF NUMERICAL SCHEMES OF VORTEX METHODS

We consider only the simplest case when the airfoil is rigid and immovable. In this case attached vortex and sources sheets intensities are equal to zero and we deal only with unknown free vortex sheet. The airfoil surface line is assumed to be closed piecewise-smooth curve \( \Gamma \), outer unit normal vector and unit tangent vector (defined on smooth parts of the curve \( \Gamma \)) we denote as \( \vec{n} \) and \( \vec{\tau} \), respectively.

As a model problem, we consider the situation when the flow domain is unbounded, there is no vorticity inside it, and the viscous incompressible incident flow has constant velocity \( \vec{V}_\infty \). Such problem statement corresponds to the instantaneous start of the flow of the medium. The problem of vortex sheet intensity computation, which should be solved at every time step at unsteady flow simulation around the airfoil, differs from the described one only in that there is known vorticity distribution in the flow domain that has an effect on the right-hand side of the corresponding integral equations.
According to the ‘classical’ approach to the boundary condition satisfaction, it is necessary to provide the equality to zero for the normal velocity component on the airfoil surface line. It leads to the integral equation of the following form \[1, 9\]:

\[
\frac{1}{2\pi} \int_{K} \left( \frac{\vec{k} \times (\vec{r} - \vec{\xi})}{|\vec{r} - \vec{\xi}|^2} \cdot \vec{n}(\vec{r}) \right) \gamma(\vec{\xi}) d\vec{\xi} = -\vec{V}_\infty \cdot \vec{n}(\vec{r}), \quad \vec{r} \in K,
\]

(1)

where \( \vec{k} \) is unit vector, which is perpendicular to the flow plane and is chosen in such way that \( \vec{k} \times \vec{n}(\vec{r}) = \vec{t}(\vec{r}) \); \( \gamma(\vec{\xi}) \) is unknown vortex sheet intensity on the airfoil surface line.

Equation (1) is singular with Hilbert-type kernel, and the principal value of the corresponding integral is considered in Cauchy sense. It has infinite set of solutions; in order to select the unique solution another additional condition is used:

\[
\int_{K} \gamma(\vec{\xi}) d\vec{\xi} = 0.
\]

(2)

The numerical schemes of vortex methods, based on approximate solution of the equations (1) and (2), we call ‘N-schemes’ (i.e., schemes derived from the boundary condition satisfaction with respect to normal velocity components).

The alternative approach to numerical scheme development is based on the equality to zero for the limit value (from the airfoil side) of flow velocity tangent component on the airfoil surface \[6, 8, 9\]

\[
\vec{V}_\infty (\vec{r}) \cdot \vec{t}(\vec{r}) = \vec{V}(\vec{r}) \cdot \vec{t}(\vec{r}) - \frac{\gamma(\vec{r})}{2} = 0, \quad \vec{r} \in K,
\]

which, in turn, leads to the integral equation of the following form

\[
\frac{1}{2\pi} \int_{K} \left( \frac{\vec{k} \times (\vec{r} - \vec{\xi})}{|\vec{r} - \vec{\xi}|^2} \cdot \vec{t}(\vec{r}) \right) \gamma(\vec{\xi}) d\vec{\xi} - \frac{\gamma(\vec{r})}{2} = -\vec{V}_\infty \cdot \vec{t}(\vec{r}), \quad \vec{r} \in K.
\]

(3)

As in the previous case, solution of this equation is not unique. In order to choose the unique solution, additional condition (2) is used as earlier.

The numerical schemes of vortex methods, based on numerical solution of the equations (3) and (2), we call ‘T-schemes’ (i.e., schemes derived from the boundary condition satisfaction with respect to tangent velocity components).

It should be noted that equations (1) and (3) are mathematically equivalent \[9\].

The kernel of the integral equation (3) is bounded if the airfoil surface \( K \) is smooth curve which belongs to \( C^2 \) class, and also if the airfoil has sharp edge which is cuspidal point (e.g., for Zhukovsky airfoils family). In the mentioned cases the solution of the equation (3) is bounded function.

It is shown, that T-schemes, i.e., schemes based on the numerical solution of the equation (3), for number of test problems provide much higher accuracy in comparison with N-schemes, based on (1) \[6, 7, 8, 9\]. In cases considered in \[6, 8\] free vortex sheet intensity was calculated for elliptical airfoils and Zhukovsky wing airfoils for both above mentioned problem statement and in presence of vorticity in the flow domain.

The choice of these airfoil becomes clear since, firstly, the kernel of the integral equation (3) for them becomes bounded, and, secondly, it is possible to obtain exact analytical solution by using conformal mapping technique \[10, 11\].
At the same time it is known [1], that in case of presence of the angle point on the airfoil surface line (with outer angle $\alpha > \pi$) vortex sheet intensity can have weak (integrable) singularity in this point, which is proportional to $\rho^{\pi/\alpha - 1}$. It means that the intensity of vortex sheet grows up when coming close to angle point and in proximity to it vortices with rather high circulations can come off the airfoil surface line. It is exactly observed in computations: main part of vorticity in vortex wake behind airfoil with angle points is generated in proximity to these angle points. Thus, the accuracy of circulations computation of such vortices determines the degree of accuracy of the whole problem solution.

In the present paper we consider the model problem of flow simulation around the airfoil with angle point and estimate the accuracy of vortex circulations calculation, generated in proximity to angle point.

3. PROBLEM STATEMENT

We consider the flow around symmetrical airfoil with length $L$, which is fully determined by the angles $\alpha$ and $\beta$, as it is shown in Fig. 1.

![Figure 1. Symmetrical airfoil with angle points](image)

The front part of the airfoil is arc of a circle having radius $R = \frac{L\sin(\beta/2)}{\sin(\beta/2) - \cos\alpha}$, its back part is formed by straight lines. The incident flow is assumed to be horizontal. As the result, the unknown vortex sheet intensity tends to infinity in the upper and lower points of the airfoil (it is assumed, that $\alpha > \pi$), and its asymptotic behavior in proximity to these points, as it follows from [1], has the following form: $\gamma \approx C\rho^{\pi/\alpha - 1}$, where $\rho$ is the distance to the angle point, $C$ is some constant.

Numerical solution of the boundary integral equation (in form (1) or (3) – it doesn’t matter) requires firstly the discretization of the airfoil surface line, that is usually done by its replacement by polygon, which legs (called ‘panels’) are straight lines. Only one vortex element is generated at every panel, as a rule. It is clear, that accuracy of numerical solution depends on number of panels, but the discretization uniformity (or, more precisely, nonuniformity) can also dramatically influence the quality of numerical solution. In particular, in $N$-schemes lengths of neighboring panels all around the airfoil should be close one to the other. It follows from the requirements for the used quadrature formula, which permits to approximate the principal value of the singular integral in (1) in Cauchy sense. In $T$-schemes on smooth parts of the airfoil the discretization can be arbitrary since the integrand in (3) is bounded function. Moreover, it should be noted, that $T$-schemes usage instead of $N$-schemes makes it possible to improve the accuracy of vortex elements circulation calculation significantly: up to 10-1000 times for
elliptic airfoils and Zhukovsky family airfoils (depending on properties of the particular airfoil and used numerical scheme) for number of panels \( N = 200 \).

Let us consider three ways to airfoil discretization into rectilinear panels and estimate the accuracy of \( N \)- and \( T \)-schemes for them. We will analyze the solution (vortex sheet intensity) on the rectilinear part of the airfoil in the neighborhood of the upper angle point. For simplicity we assume that the arc part of the airfoil is split into panels of equal length and consider the following ways to rectilinear part discretization:

- panels have equal lengths which is close as possible to panel length on arc part;
- panels have equal lengths which differ significantly from panel length on arc part;
- panels have different lengths, which form, for example, geometric series; panels in proximity to upper angle point are smaller then the panels far from it.

4. RESULTS OF NUMERICAL SIMULATION FOR THE AIRFOIL WITH UNIFORM SURFACE LINE DISCRETIZATION

The detailed description of number of numerical schemes for \( N \)- and \( T \)-approaches and technique for their accuracy estimation can be found in [6, 7, 8, 11]. Here we only note, that the most obvious way for comparison of the numerical solution for vortex sheet intensity with the exact one (or at least, with reference one) is incorrect in the considered case due to unboundedness of the solution in proximity to the angle point. However, vortex sheet intensity has sense of ‘circulation density’, while the integral value, namely circulation itself has the physical sense. So the correct way is to compare circulations of vortices which come off the airfoil surface line and form vortex wake behind it. They are calculated as total vorticity values over some parts of the airfoil. In \( T \)-schemes, we consider the panels of the airfoil as such parts, in the framework of \( N \)-schemes, circulation is integrated over two halves of adjoining panels with common vertex.

In the considered case the exact analytical solution can’t be derived, however its asymptotic behavior in neighborhood of the angle point is known, so it is possible to suggest the following technique. For the given airfoil with known parameters \( L, \alpha, \beta \) and known incident flow velocity \( V_\infty \) we solve the problem numerically for different values of number of panels \( N \). Both \( N \)- and \( T \)-schemes are used; the airfoil surface line is assumed to be split into panels uniformly (i.e., all the panels have the same lengths).

Taking into account that solution in neighborhood of the angle point satisfies dependence \( \gamma'(\rho) = C \rho^{\pi/(\alpha - 1)} \), it is possible to calculate average values of vortex sheet intensity for \( n_p \) nearest neighbor panels to the angle point:

\[
\Gamma_i^* = \int_{\rho_i}^{\rho_{i+1}} \gamma'(\rho) d\rho = \int_{\rho_i}^{\rho_{i+1}} C \rho^{\pi/(\alpha - 1)} d\rho = \frac{\alpha C}{\pi} (\rho_{i+1}^{\pi/(\alpha - 1)} - \rho_i^{\pi/(\alpha - 1)}) + \Gamma_i, \quad i = 1, 2, \ldots, n_p.
\]

Here \( \rho_i \) is the distance from the angle point to the beginning of the \( i \)-th panel.

The \( T \)-scheme permits directly to determine values \( \gamma_i \) – average vortex sheet intensities over the panels, so it is possible to write down the following expression:

\[
\gamma_i = \frac{\Gamma_i^*}{\rho_{i+1} - \rho_i} = \frac{\alpha C}{\pi} \frac{\rho_{i+1}^{\pi/(\alpha - 1)} - \rho_i^{\pi/(\alpha - 1)}}{\rho_{i+1} - \rho_i}.
\]
Then we summarize the squared differences between calculated and ‘asymptotic’ average values

$$\Psi(C) = \sum_{i=1}^{n_p}(\gamma_i - \gamma_i^*)^2,$$

where summation is performed for $n_p$ nearest to the angle point panels; in the present paper, it is assumed that $n_p = 5$. The value of constant $C$ can be found from minimization condition for the function $\Psi(C)$.

In case of $N$-scheme, the technique remains nearly the same but with the only difference that the value $\gamma_i$ which corresponds to the average value of vortex sheet intensity is computed over two halves of the adjoining panels, namely the panels having indices $i$ and $(i+1)$. Therefore the asymptotic solution also should be integrated and averaged over the lines with ends at points $\frac{\rho_i + \rho_{i+1}}{2}$ and $\frac{\rho_{i+1} + \rho_{i+2}}{2}$.

The results of the coefficient $C$ reconstruction for different number of panels on the arc part of the airfoil and the corresponding number of panels on the whole airfoil are given in Table 1.

<table>
<thead>
<tr>
<th>Num. of panels on the one half of the arc part</th>
<th>Total number of panels</th>
<th>$C_N$</th>
<th>$C_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>180</td>
<td>0.8081</td>
<td>0.8119</td>
</tr>
<tr>
<td>40</td>
<td>360</td>
<td>0.8016</td>
<td>0.8072</td>
</tr>
<tr>
<td>60</td>
<td>538</td>
<td>0.7991</td>
<td>0.8058</td>
</tr>
<tr>
<td>80</td>
<td>718</td>
<td>0.7983</td>
<td>0.8052</td>
</tr>
<tr>
<td>100</td>
<td>896</td>
<td>0.7977</td>
<td>0.8049</td>
</tr>
<tr>
<td>150</td>
<td>1344</td>
<td>0.7972</td>
<td>0.8046</td>
</tr>
<tr>
<td>200</td>
<td>1792</td>
<td>0.7970</td>
<td>0.8045</td>
</tr>
<tr>
<td>300</td>
<td>2688</td>
<td>0.7970</td>
<td>0.8045</td>
</tr>
</tbody>
</table>

It is seen that the larger is number of the panels, the closer value of $C$ to some values. The difference in estimations for $C$ obtained by using $N$- and $T$-schemes is smaller than 1%. This gap can be caused by number of factors, in particular, by the specific features of the numerical schemes, and (which seems to be more important) by the fact, that the solution doesn’t coincide exactly with the asymptotic one and can be written down in the following form

$$\gamma_{exact}(\rho) = C\rho^{\pi/a-1} + h(\rho),$$

where $h(\rho)$ is some unknown bounded function, which, however, haven’t been taken into account at computation of $C$ value.

Note, that in the same problem for rhombic airfoil, which back part is the same as the back part of the airfoil shown in Fig. 1, and its front part is obtained by symmetrical reflection, the solution singularity has the same asymptotics, and the values of the coefficient $C$ obtained by using $N$- and $T$-schemes differ less than 0.1%.

In Fig. 2 for total number of panels $N = 896$ the numerical average values for vortex sheet intensities $\gamma_i$ are shown (points) for some nearest neighbor to the angle point panels of the airfoil in comparison with average values of the asymptotic solution $\gamma_i^*$ calculated for the above
obtained value of $C$ (the corresponding points are joined by solid line). The ‘visual difference’ between results for $N$- and $T$-schemes is connected with the different choice of the parts of the airfoil for solution averaging, what has been discussed earlier.

![Figure 2](image.png)

**Figure 2** Computed values (dots) of the average vortex sheet intensity in the neighborhood of the angle point in comparison to average values of the asymptotic solution (solid line) for the $N$-scheme ($a$) and $T$-scheme ($b$)

The average values for the relative deviations of computed values of average vortex sheet intensity against ones for asymptotic solution are given in Table 2. The averaging was performed for $n_p = 5$ nearest to the angle point neighbor parts of the airfoil for which computations were carried out. Recall, that all computations are carried out for uniform airfoil splitting into panels.

### Table 2. Deviations of the computed values of the average vortex sheet intensity against asymptotic solution

<table>
<thead>
<tr>
<th>Number of panels</th>
<th>180</th>
<th>360</th>
<th>538</th>
<th>718</th>
<th>896</th>
<th>1344</th>
<th>1792</th>
<th>2688</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_N$, %</td>
<td>2.41</td>
<td>1.56</td>
<td>1.26</td>
<td>1.12</td>
<td>1.04</td>
<td>0.95</td>
<td>0.92</td>
<td>0.90</td>
</tr>
<tr>
<td>$\delta_T$, %</td>
<td>2.12</td>
<td>1.36</td>
<td>1.12</td>
<td>1.00</td>
<td>0.94</td>
<td>0.80</td>
<td>0.85</td>
<td>0.83</td>
</tr>
</tbody>
</table>

It is clear, that both $N$- and $T$-schemes permit to compute vortex sheet intensity with rather high accuracy for the airfoil with angle point. This conclusion, in fact, is true only for the cases which are close to the considered one in sense of vorticity absence in the flow in proximity to the airfoil surface line. The accuracy estimation for the different numerical schemes in presence of vortex elements in the flow is non-trivial problem; it is partially considered (for circular airfoil) in [8].

5. RESULTS OF NUMERICAL SIMULATION FOR THE AIRFOIL WITH NON-UNIFORM SURFACE LINE DISCRETIZATION

In some cases in practice it is not easy to provide the equal lengths of the panels on the airfoil surface line, in particular, the airfoil can be ‘constructed’ from separate curves with preliminary introduced discretization into the panels. The same refers to flow simulation around deformable airfoils, when panel length are naturally change during the simulation process.

For the above described problem we firstly consider the case when both arc and rectilinear parts of the airfoil (Fig. 1) are split into panels uniformly, but the lengths of panels on these parts are different. Let us assume, that the arc part is split into 200 panels, while number of panels on rectilinear parts will be varied, specifying the length ratio $\kappa > 1$ for panel length adjoining to the angle point. In order to estimate accuracy of numerical solutions, we use as
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the reference the previously obtained asymptotic solution with the corresponding coefficient $C$ value.

In Table 3 the average values for the relative deviations of computed values of the average vortex sheet intensity against ones for asymptotic solution are given for the described airfoil discretization. The averaging, as earlier, was performed for $n_p = 5$ nearest panels to the angle point.

Table 3 Deviations of the computed values of the average vortex sheet intensity against asymptotic solution at different panel length on arc and rectilinear parts of the airfoil

<table>
<thead>
<tr>
<th>Panel length ratio $\kappa$</th>
<th>1.10</th>
<th>1.25</th>
<th>1.33</th>
<th>1.50</th>
<th>2.00</th>
<th>3.00</th>
<th>5.00</th>
<th>10.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number of panels</td>
<td>966</td>
<td>1070</td>
<td>1128</td>
<td>1244</td>
<td>1592</td>
<td>2288</td>
<td>3680</td>
<td>7158</td>
</tr>
<tr>
<td>$\delta_N, %$</td>
<td>1.37</td>
<td>2.74</td>
<td>3.49</td>
<td>4.93</td>
<td>8.95</td>
<td>15.92</td>
<td>27.21</td>
<td>47.65</td>
</tr>
<tr>
<td>$\delta_T, %$</td>
<td>0.98</td>
<td>1.06</td>
<td>1.10</td>
<td>1.19</td>
<td>1.45</td>
<td>1.92</td>
<td>2.53</td>
<td>3.13</td>
</tr>
</tbody>
</table>

It is clearly seen, that the error for both schemes becomes higher as panel length ratio growth, however for rather high values of $\kappa$ the $T$-scheme provides approximately 10 times higher accuracy in comparison to the $N$-scheme. The results of numerical solution of the problem at $\kappa = 10.0$ are shown in Fig. 3.

Figure 3 Computed values (dots) of the average vortex sheet intensity in the neighborhood of angle point at $\kappa = 10.0$ in comparison to average values of the asymptotic solution (solid line) for the $N$-scheme ($a$) and $T$-scheme ($b$)

Numerical simulations show that the uniformity of the rectilinear part of the airfoil discretization doesn’t influence significantly the result. In particular, we consider quasi-uniform discretization when panel lengths form geometric series with ratio $q = 0.95 \ldots 0.99$ (that corresponds to mesh refinement in neighborhood of the upper angle point). If we consider the same number of panels as earlier in Table 3 on the arc part of the airfoil and the same $\kappa$ value for the panels ratio which adjoin the angle point, the final results remain nearly the same. For the $T$-scheme deviation from the asymptotic solution practically doesn’t change, for the $N$-scheme the error becomes slightly larger and is equal to 54 % at $q = 0.95, \kappa = 10$. However in the last case total number of panels is only 400 (200 on the arc part, 200 on two rectilinear segments), while it was equal to 7158 for uniform splitting (Table 3).

At ‘more non-uniform’ airfoil surface line discretization ($\kappa > 10$), relative error of the $N$-scheme becomes higher and the numerical solution becomes qualitatively wrong, while the error of the $T$-scheme even in this case remains acceptable and the obtained solution is qualitatively correct. The results for such case, when number of panels on the arc part of the circle is equal to 200 and geometric series ratio is $q = 0.98$, for large values of $\kappa$ ratio are given in Table 4.
Table 4 Deviations of the computed values of the average vortex sheet intensity against asymptotic solution at large $\kappa$ ratio of panel length adjoining to the angle point

<table>
<thead>
<tr>
<th>Number of panels on rectilinear part</th>
<th>450</th>
<th>500</th>
<th>550</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel length ratio $\kappa$</td>
<td>13.1</td>
<td>21.9</td>
<td>36.3</td>
<td>60.3</td>
</tr>
<tr>
<td>Total number of panels</td>
<td>650</td>
<td>700</td>
<td>750</td>
<td>800</td>
</tr>
<tr>
<td>$\delta_N$, %</td>
<td>69.6</td>
<td>82.0</td>
<td>106.8</td>
<td>135.1</td>
</tr>
<tr>
<td>$\delta_T$, %</td>
<td>3.7</td>
<td>5.6</td>
<td>8.3</td>
<td>11.5</td>
</tr>
</tbody>
</table>

6. THE AIRFOIL WITH SMALLER THAN STRAIGHT OUTER ANGLE

Let us consider also the case of the airfoil with angle point with outer angle smaller than straight angle $\alpha < \pi$ (Fig. 4). The algorithm for such airfoil specification remains the same as earlier.

![Figure 4 The airfoil with angle points](image)

It is mentioned in [1] that in such angle points vortex sheet intensity has no singularity and it is bounded. At uniform airfoil discretization, when all the panels on the surface line have nearly the same length, both $N$- and $T$-schemes permit to obtain qualitatively correct solution. At non-uniform splitting for $\kappa < 2$ (the same notation is used as in the previous section) solution also remains qualitatively correct and practically doesn’t changes quantitatively.

The larger values of $\kappa$ (i.e., ‘strong non-uniform’ discretization) lead to significant error of numerical solution obtained by using the $N$-scheme, while the $T$-scheme provides the solution which is close to original one. In Fig. 5 the dependence of the average vortex sheet intensity against the distance to angle point is shown being measured along the rectilinear part of the airfoil. The solid line corresponds to the solution obtained for very fine uniform surface line discretization into $N \approx 4000$ panels, dots denote the results obtained for non-uniform splitting; values of $N$ and the corresponding values of $\kappa$ were varied, computations were carried out for $N$- and $T$-scheme. The arc part of the airfoil in all cases was split into approximately 200 panels, panel lengths in rectilinear part formed geometric series with ratio $q = 0.98$ (with refinement in neighborhood of the angle point).
**Figure 5** Computed values (dots) of the average vortex sheet intensity in the neighborhood of angle point at $N = 440, \kappa \approx 2.81$ (a, b); $N = 500, \kappa \approx 5.39$ (c, d); $N = 600, \kappa \approx 15.27$ (e, f) in comparison with solution obtained for very fine uniform discretization (solid lines). The computations are carried for the $N$-scheme (a, c, e) and the $T$-scheme (b, d, f).

**7. CONCLUSION**

The results of the model problem solution for flow simulation around the airfoil with angle point by using $N$- and $T$-schemes of vortex methods in the absence of vorticity in the flow domain show that both approaches are applicable and provide rather high accuracy in case of uniform airfoil discretization. At non-uniform discretization when panels adjoining the angle point have significantly different lengths, the $T$-scheme provides much more accurate solution in comparison to the $N$-scheme. If the panel ratio exceeds 10, the qualitatively correct solution can not be obtained by using the $N$-scheme while the $T$-scheme provides qualitatively correct solution and, moreover, the error remains acceptable for number of applications.

Nearly the same results are obtained both for airfoils with ‘outer’ angle point where the solution has weak singularity, and for airfoils with ‘inner’ angle point where solution is bounded.

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