EQUATION FOR RESILIENCE IN A LINEARLY TAPERING SOLID SHAFT OF CIRCULAR CROSS SECTION SUBJECTED TO TORSION

B Vishwash
Assistant Professor, Department of Mechanical Engineering,
Siddaganga Institute of Technology, Tumakuru, Karnataka, India

ABSTRACT

In this paper a new equations have been developed to obtain the resilience for linearly tapering shaft subjected to pure torsion. The strain energy equations have been developed first and then the same are used to develop the resilience equations. One strain energy equation and the corresponding resilience is obtained by suitably neglecting the terms and considering the average volume. Final resilience equation is developed by considering the original volume.

Key words: Torsion, Angle of twist, Strain Energy, Resilience and Proof resilience.

AMS Classification: 05C07, 68R10, 03E72

http://www.iaeme.com/IJME/issues.asp?JType=IJMET&VType=9&IType=2

1. INTRODUCTION

When a member is subjected to moment about it axis then it is considered to be under torsion. The torsional moment is also called as torque or twisting moment because the effect of torsional moment on the member is to twist the member. In the field of engineering majority of members are subjected torsion. Some of the example include shafts transmitting power from a motor to machine, from engine to the rear axle of automobile, from a turbine to electric motors, ring beam of circular water tanks and beams of grid flooring system. If the cross sections of members are subjected to only torsional moments and not accompanied by bending moment and axial forces, then the member is under pure torsion. While developing the theory of pure torsion assumption made are the material is assumed to be homogeneous and isotropic, the stresses developed are assumed to be within the elastic limit, cross sections are plane before applying twisting moment are assumed to remain plane even after the application of twisting moment (no warping takes place), radial lines are assumed to remain radial even after applying torque and the twist along the shaft is assumed as uniform.
2. STRAIN ENERGY IN A LINEARLY TAPERING SHAFT

When a torque $T$ is applied to a linearly tapering solid shaft, it gets twisted by an angle $\theta$. Thus the twisting moment does the work on the shaft and this work done is stored as strain energy in the shaft. When $T$ is applied gradually the angle of rotation increases linearly and reaches the value of $\theta$. From the torsion equation, for a given shaft the relationship between $T$ and $\theta$ is linear as shown in Figure 1.

![Figure 1 Relationship between torque and angle of twist (linear relationship)](image)

**Strain Energy** = $U = \text{Area under the curve}$

$$U = \frac{1}{2} \times \text{Base} \times \text{Height}$$

$$U = \frac{1}{2} \times \theta \times T$$

From the equation of torsion we have the angle of twist as

$$\theta = \frac{T \times L}{G \times J}$$

where, $T = \text{Torque}$

$L = \text{Length}$

$G = \text{Modulus of rigidity or shear modulus}$

$J = \text{Polar moment of inertia}$

Therefore strain energy is given by, we get

$$U = \frac{1}{2} \times \left(\frac{T \times L}{G \times J}\right) \times T$$

$$U = \frac{1}{2} \times \left(\frac{T^2 \times L}{G \times J}\right)$$

Consider a linearly tapering solid shaft having circular cross section, tapering from diameter $d_2$ to $d_1$ as shown in Figure 2.
Rate of change of diameter $= \frac{d_2 - d_1}{L}$

$\therefore$ Diameter at a distance $y$ from the end of a bar of diameter $d_1$,

$$d = d_1 + \left(\frac{d_2 - d_1}{L}\right)y$$

Polar moment of inertia for circular cross section, $J = \frac{\pi}{32} \times d^4$

$\therefore J = \frac{\pi}{32} \times \left[d_1 + \left(\frac{d_2 - d_1}{L}\right)y\right]^4$

Also, from the equation of torsion we have torque as,

$$T = \frac{\tau \times J}{r}$$

where, $T =$ Torque
$r =$ Radius or distance of element from center of shaft
$\tau =$ Shear stress in the element at radius $r$
$J =$ Polar moment of inertia

$$T = \frac{\tau \times J}{\left(\frac{d}{2}\right)} = \frac{2\tau \times J}{d}$$

$$T = \frac{2\tau \times \left\{\frac{\pi}{32} \times \left[d_1 + \left(\frac{d_2 - d_1}{L}\right)y\right]^4\right\}}{d_1 + \left(\frac{d_2 - d_1}{L}\right)y}$$

$$T = \tau \times \left\{\frac{\pi}{16} \times \left[d_1 + \left(\frac{d_2 - d_1}{L}\right)y\right]^3\right\}$$

On substituting polar moment of inertia and torque equations in strain energy equation, we have
Equation for Resilience in a Linearly Tapering Solid Shaft of Circular Cross Section Subjected to Torsion

\[
U = \frac{1}{2} \left\{ \frac{\tau \pi}{16} \left[ d_1 + \left( \frac{d_2 - d_1}{L} \right) y \right]^3 \right\}^2 \times L
\]

\[
U = \frac{1}{2} \left\{ \frac{\tau^2 \pi^2 \times 32 \times G \times \pi \times 16^2}{d_1 + \left( \frac{d_2 - d_1}{L} \right) y} \right\}^6 \times (L)
\]

\[
U = \frac{\tau^2 \pi L}{16G} \times \left[ d_1 + \left( \frac{d_2 - d_1}{L} \right) y \right]^2
\]

\[
U = \frac{\tau^2 \pi L}{16G} \times \left\{ d_1^2 + \left( \frac{d_2 - d_1}{L} \right)^2 + 2 \times d_1 \times \left( \frac{d_2 - d_1}{L} \right) y \right\}
\]

\[
U = \frac{\tau^2 \pi L}{16G} \times \left\{ d_1^2 + \left( \frac{d_2 - d_1}{L} \right)^2 \times \left( \frac{y}{L} \right)^2 \right\} + 2 d_1 d_2 \left( \frac{y}{L} \right) - 2 d_1^2 \left( \frac{y}{L} \right)^2
\]

\[
U = \frac{\tau^2 \pi L}{16G} \times \left\{ d_1^2 \left[ 1 + \left( \frac{y}{L} \right)^2 \right] - 2 \times 1 \times \left( \frac{y}{L} \right) + d_2^2 \left( \frac{y}{L} \right)^2 + 2 d_1 d_2 \left( \frac{y}{L} \right) - 2 d_1^2 \left( \frac{y}{L} \right)^2 \right\}
\]

\[
U = \frac{\tau^2 \pi L}{16G} \times \left\{ d_1^2 \left( 1 - \frac{y}{L} \right)^2 + d_2^2 \left( \frac{y}{L} \right)^2 + 2 d_1 d_2 \left( \frac{y}{L} \right) \times \left[ 1 - \frac{y}{L} \right] \right\}
\]

\[
\therefore \text{Strain Energy, } U = \frac{\tau^2 \pi L}{16G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2
\]

By neglecting the terms \( \left( \frac{y}{L} \right) \) in above equation we get,

\[
U = \frac{\tau^2 \pi L}{16G} \times (d_1 + d_2)^2
\]

\[
U = \frac{\tau^2 \pi L}{16G} \times \left( \frac{d_1 + d_2}{2} \right)^2
\]

\[
U = \frac{\tau^2 \pi L}{8G} \times \left( \frac{d_{\text{avg.}}}{2} \right)^2
\]

where, \( d_{\text{avg.}} \) = average diameter of a linearly tapering shaft = \( \frac{d_1 + d_2}{2} \)

\[
U = \frac{\tau^2 \pi}{8G} \times \left[ L \times \frac{\pi}{4} \times (d_{\text{avg.}})^2 \right]
\]
3. RESILIENCE IN THE LINEARLY TAPERING SHAFT

The strain energy per unit volume is called as resilience. The maximum strain energy per unit volume, which can be stored by a body without undergoing permanent deformation is known as proof resilience. Proof resilience is strain energy per unit volume in the body corresponding to stress at elastic limit.

3.1. Considering Average Volume

Consider the strain energy equation of linearly tapering shaft developed above as

\[ U = \frac{\tau^2}{G} \times (L \times A_{avg.}) \]

where, \( A_{avg.} = \) average area of a linearly tapering shaft = \( \frac{\pi}{4} \times (d_{avg.})^2 \)

\[ U = \frac{\tau^2}{G} \times V_{avg.} \]

where, \( V_{avg.} = \) average volume of a linearly tapering shaft = \( V_{avg.} \).

\[
\text{Strain Energy} = \frac{(\text{Shear stress})^2}{\text{Shear Modulus}} \times \text{Average volume of a linearly tapering shaft}
\]

3.2. Considering Original Volume

Consider the strain energy equation of linearly tapering shaft developed above as

\[ U = \frac{\tau^2}{G \times 16} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2 \]
This equation of strain energy is obtained by considering the original volume of the linearly tapering shaft. By considering this equation, equation of resilience can be developed as below.

**Resilience, \( R \) = Strain energy per unit volume

\[
Resilience, \ R = \frac{\text{Strain energy}}{\text{Volume}}
\]

\[
Resilience, \ R = \frac{\frac{\tau^2 \pi L}{16G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2}{V}
\]

Since the original volume of the linearly tapering shaft is to be considered, the volume will be obtained by multiplying the length to the integrated area between two diameters. Hence the resilience equation is given as below for the linearly tapering shaft considered the original volume \( V \).

\[
Resilience, \ R = \frac{\frac{\tau^2 \pi L}{16G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2}{V}
\]

\[
R = \frac{\frac{\tau^2 \pi L}{16G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2}{\int_{d_1}^{d_2} \left( \text{Length} \times \text{Area} \right) d\alpha}
\]

\[
R = \frac{\frac{\tau^2 \pi L}{16G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2}{\int_{d_1}^{d_2} \left( L \times \frac{\pi}{4} \times d^2 \right) d\alpha}
\]

\[
R = \frac{\frac{\tau^2 \pi L}{16G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2}{L \times \frac{\pi}{4} \int_{d_1}^{d_2} (d^2) d\alpha}
\]

\[
R = \frac{\tau^2}{4G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2
\]

\[
\left[ \int_{d_1}^{d_2} (d^2) d\alpha \right]
\]

\[
R = \frac{\tau^2}{4G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2
\]

\[
\left[ \int_{d_1}^{d_2} (d^2) d\alpha \right]
\]

\[
R = \frac{\tau^2}{4G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2
\]

\[
\left[ \int_{d_1}^{d_2} (d^2) d\alpha \right]
\]

\[
R = \frac{3\tau^2}{4G} \times \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2
\]

\[
\left( d_2^3 - d_1^3 \right)
\]

\[
\therefore \ Resilience, \ R = \frac{3\tau^2}{4G(d_2^3 - d_1^3)} \times \int_0^L \left[ d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right) \right]^2 dy
\]
\[ R = \frac{3\tau^2}{4G(d_2^3 - d_1^3)} \times \left[ \frac{d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right)^3}{3} \right]_0^L \times \frac{1}{\left( \frac{d_2}{L} - \frac{d_1}{L} \right)} \]

\[ R = \frac{\tau^2 L}{4G(d_2^3 - d_1^3)(d_2 - d_1)} \times \left[ \frac{d_1 \left( 1 - \frac{y}{L} \right) + d_2 \left( \frac{y}{L} \right)^3}{3} \right]_0^L \]

\[ R = \frac{\tau^2 L}{4G(d_2^3 - d_1^3)} \times \left( d_2^2 - d_1^2 \right) \]

\[ R = \frac{\tau^2 L}{4G(d_2 - d_1)} \]

\[ \therefore \text{Resilience for linearly tapering circular shaft, } R = \frac{\tau^2 L}{4G(d_2 - d_1)} \]

4. CONCLUSIONS

The two strain energy equations and the corresponding resilience equations have been developed for the linearly tapering shaft of circular shaft. The resilience equation developed from the strain energy by neglecting term \( \left( \frac{y}{L} \right) \) and considering original volume. This equation reveals that resilience is independent of the dimensions of the shaft (length and cross sections) and only dependent on the applied torque and shear modulus and varies only with load. Another resilience equation developed from the strain energy equation by considering original volume without any assumptions and neglecting of terms reveals that the resilience is dependent not only on the applied torque and shear modulus but also on the dimensions of the shaft namely length and diameters. The major outcome of the resilience equation developed for the linearly tapering shaft of circular cross section is that, the resilience value is inversely proportional to the tapering angle if length, shear modulus and applied torque are kept constant.

REFERENCES


http://www.iaeme.com/IJMET/index.asp editor@iaeme.com
Equation for Resilience in a Linearly Tapering Solid Shaft of Circular Cross Section Subjected to Torsion

Sciences (ICAMS 2017), Department of Mechanical Engineering, Malnad College of Engineering, Hassan, March 3rd to 5th 2017, Page No. - 47.


