A JOINT TWO-ECHELON INVENTORY MODEL COUPLED WITH CREDIT PERIOD AND QUADRATIC PRICE DEPENDENT DEMAND

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ABSTRACT

In the current global business competitive environment, in order to get the win-win situation, the need for coordination mechanism become inevitable. This paper characterizes the two-echelon inventory system under credit period mechanism and quadratic price dependent demand. A supplier supplying a single kind of product to a buyer is considered in the inventory system. Optimal cycle time, number of shipment deliveries from a supplier to a buyer are considered as decision variables. Mathematical models are formulated for annual total variable cost of the inventory at the supplier node, buyer node and then for the supply chain. The objective of the model is to illustrate inventory associated decisions under credit period and quadratic price dependent demand. Analytical proofs are derived to prove the convexity of the model. A case study data devised, computer program is developed in MATLAB and the model is solved. Sensitivity analysis is also carried out to infer the managerial decisions.

Keywords: Credit period mechanism; Quadratic price dependent demand; Annual total variable cost; Inventory associated decisions;

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1. INTRODUCTION
The last decade of the twentieth century was a period of rapid change for business organisations. Further, it is a fact that the rate of change in business competition has been increasing in the twenty-first century. Consequently, manufacturing/business organizations realized to transform themselves to survive in the intensely competitive global environment. Also, it has become evident to understand the complexities in the global environment, anticipate significant changes, and adapt to those changes as needed.

Developing organizational capability and infrastructure to manage supply chain activities on a global basis is an increasingly important challenge for supply chain management. Further, supporting of globalization strategies of the firm, represents a significant and important issue for SCM.

In present day contest of inherited business world, overseeing inventory is a key dispute in all the economy sectors. The product demand plays a great role while overseeing the goods stock, as it may depend on several elements like selling price of the product, publicity of goods, inflation etc. Out of all these elements, product selling price is a crucial element for the business, since in all the business competitive world, one is frequently lookout to beat the competition for the ways. To improve the aggregate demand of the product, more consumers need to get attracted by maintaining a reasonable price in the market. To improve business operations, the efficiency of the organization depends on the effective supply chain management. Hence coordination plays a vital role in the supply chain. He at al. [1] proposed inventory, manufacturing, and contracting of a multi-echelon supply chain decisions with both criteria in product demand and uncertainties in supply.

The authors are concentrated on a return policy with supplier and buyer along with whole price discount. Mitra et al [2] investigated the effect of demand utilising information in a single-ware house and two vendors system under periodic review. A general and predetermined replenishment interval is assumed for all intermediate stages. Deterministic lead times are compared with replenishment interval. The ware house shortages can be considered as two criteria—1. emergency rate of shipment and 2. allocation. Gumus et al [3] had developed the effective and efficient multi echelon supply chain with fuzzy and stochastic environments. By the frame work of the authors, realistic cost and efficient forecast can be evaluated in their proposed model. Cheng et al [4] had investigated the effect of regulations of carbon emission on inventory routing problem (IRP). The author presented model related to conventional IPR model, which consists of inbound products collections system with one assembly plant and set of geologically distributed suppliers. To evaluate the overall variable cost of the system, the author assumed that the transportation cost is fixed and considered the inventory holding cost and fuel expenditure cost. The organization of the present paper is as follows; in Section 2 detailed literature review is framed. In Section 3, described about a mathematical modelling of the problem with some set of assumptions and notations, and Section 4 consists of numerical illustration. Finally, in Section 5 presented results and conclusions along with the future scope of the present model.

2. LITERATURE REVIEW
It is extremely necessary to plan and control a supply chain system in such a way that the total costs are reduced in addition to have more challenging with system wise service levels at different stages. In the review of literature, different inventory policies for supply chain collaboration have been introduced. For example, Zhao et al [5] proposed that an integrated manufacturing-inventory allocation planning problem in an infinite horizon time. The authors had developed a best numerical-proportion coordinated policy for replenishment inventory throughout its supply chain. Schmitt et al [6] had analysed the risk of supply chain with the
inventory position and back-up methods in a multi-echelon networks system. Al Hanbaliet al [7] investigated interval time of a two-echelon multi product spare parts inventory control system. Authors modelled the entire system as a Markov chain and analyzed the problem under finite time interval.

Haji et al [8] proposed a two echelon inventory system under a new ordering policy with a single warehouse and a huge number of non-identical vendors. In their paper, the warehouse implements modifications in one for one policy. Al-Rifai et al [9] presents a two-echelon non-repairable spare parts inventory control system consisting of single warehouse and m identical vendors and measured the replenishment point, quantity ordered (R,Q) inventory policy. Zahraei et al [10] considered the manufacturing smoothing and its consequences on inter-stage flow of demand in a complex supply chain. Alikar et al [11] studied multi-item multi-time interval series-parallel inventory redundancy allocation problem (SPIRAP). The objective of the authors is to study the optimal replenishment quantity of the items for each and every subsystem in each time interval period.

Wang et al [12] proposed the model related to a two-stage manufacturing capacity with regular probability and stochastic breakdowns. AlDurgam et al [13] proposed a model with the coordination of both manufacturing and allocation of the items in such way that the entire supply chain costs are reduced. Shastri et al [14] framed a multi-echelon inventory system for deteriorating products with one-sided backlogging under inflationary situation. Hoseinia et al [15] proposed an inventory management system in a multi-channel allocation system consisting of a one producer, with a one kind of item, and a random number of vendors with a stochastic demand. Pal et al [16] framed a defective EPQ inventory system in price dependent demand over two cycles. Qi et al [17] proposed make-to-order supply chain having a single seller and two buyers in a two-echelon inventory system under carbon cap regulation. The author formulated decentralized system from a theoretical game perspective in order to obtain pricing decision. Sarkar [18] developed a model for deteriorating products in a two echelon inventory supply chain system. Qin et al [19] studied the influence of inexact inventory decisions on bullwhip effect under the provisions of order variation.

Most of the researchers are concentrated on more about the two echelon inventory system models with some set of assumptions. This study made an attempt to establish a two echelon inventory systems under credit period with quadratic price dependent demand, where single supplier is supplying one kind of product to the one buyer. In the proposed model, product demand is a quadratic function of retailer unit selling price. The concept of trade credit is considered as a coordination mechanism between supplier and buyer. Whenever goods are received by the buyer from the supplier, the buyer can delay the payments. The supply chain cost is expressed in terms of annual ordering cost, carrying cost along with transportation cost of respective entities. The main objective of the work is to show the optimal replenishment quantities at buyer, number of shipments between supplier and buyer, and annual total relevant cost of supply chain under the quadratic price dependent demand.

### 3. MATHEMATICAL MODELLING

In the suggested model development, a two-echelon inventory system, encompassing a single kind of supplier shipping a single kind of product to a single buyer, is examined.

#### 3.1. Features and assumption

1.1. For the formulation of mathematical model, the following features and assumptions are considered.

a) Demand is a quadratic function of buyer unit selling price

b) Infinite rate of production
c) Instantaneous replenishment rate
d) Supplier (may be a Manufacturer) inventory level is an integer multiple of buyer ordering quantity
e) Shortages are not allowed
f) Supplier provides the trade credit to the Buyer

3.2 Notation

The following notations are used in development of the mathematical model.

\( \omega \) Demand rate of the buyer per year (in units/annum)
\[ \therefore \omega = \left( \alpha - \beta R_p - \gamma R_p^2 \right) \beta > 0, \alpha >> \gamma \text{ and } \alpha << \beta \]

\( R_A \) Buyer ordering cost (in INR/order)
\( M_A \) Supplier fixed cost of setup (in INR/lot)
\( R_C \) Buyer unit cost (in INR/unit)
\( M_C \) Supplier unit cost (in INR/unit)
\( P_R \) Buyer unit selling price of the product (in INR/unit)
\( M_s \) Supplier shipment cost to deliver goods to buyer (in INR/delivery)
\( R_t \) Buyer shipping cost for accepting delivery from supplier (in INR/shipment)
\( H_e \) Interest earned (in INR/INR/year)
\( H_p \) Interest paid (in INR/INR/year)
\( e \) Credit period
\( q \) Quantity of delivered goods from supplier to buyer in each delivery (in units)
\( \omega \) Number of deliveries from supplier to the vendor
\( E \) The cycle time length at buyer (in years)
\( R_p \) Annual overall cost of buyer (in INR)
\( M_q \) Annual overall cost of supplier (in INR)
\( S_q \) Annual overall cost of supply chain (in INR)

3.3. Model Formulation

In the formulation of a mathematical model, it is assumed that the supplier provides trade credit to the buyer. Various inventory associated cost factors like ordering cost, carrying cost and transportation costs incurred at the buyer and supplier are included in the development of the model as shown below.

3.1.1. Buyer Point

**Case I: \( E \geq e \)**

Cost of replenishing inventory at the buyer per annum is \( \frac{R_A}{E} \)

Cost of holding inventory at the buyer per annum is \( \frac{(\alpha - \beta R_p - \gamma R_p^2) ER_p}{2} \)

\[ \therefore \omega = \left( \alpha - \beta R_p - \gamma R_p^2 \right) \]
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Transportation cost incurred at buyer to receive the shipment from supplier per annum is expressed as \( \frac{R_c}{E} \)

Interest rate paid by the buyer per cycle is
\[
\frac{R_c \left(\alpha - \beta R_p - \gamma R_p^2\right) (E - e)^2 H_p}{2}
\]

Interest rate paid by the buyer per annum is
\[
\frac{R_c \left(\alpha - \beta R_p - \gamma R_p^2\right) (E - e)^2 H_p}{2E}
\]

Interest rate earned by the buyer per annum is
\[
\frac{R_p \left(\alpha - \beta R_p - \gamma R_p^2\right) e^2 H_e}{2E}
\]

Annual overall cost of buyer is obtained by subtracting interest earned per year from the sum of annual cost of ordering, cost of holding, cost of shipping and interest paid per annum.
\[
R_p(E) = \frac{R_A}{E} + \frac{\left(\alpha - \beta R_p - \gamma R_p^2\right) E R_c H_p}{2} + \frac{R_c \left(\alpha - \beta R_p - \gamma R_p^2\right) (E - e)^2 H_p}{2E} - \frac{R_p \left(\alpha - \beta R_p - \gamma R_p^2\right) e^2 H_e}{2T}
\]

Case II: \( e > E \)

Interest earned by the buyer per annum is
\[
R_p \left(\alpha - \beta R_p - \gamma R_p^2\right) H_e \left(e - \frac{E}{2}\right)
\]

Annual overall cost of buyer is obtained by subtracting the interest earned per annum from the sum of cost of ordering, cost of holding, cost of shipping and interest paid per annum.
\[
R_p(E) = \frac{R_A}{E} + \frac{R_c}{E} + \frac{\left(\alpha - \beta R_p - \gamma R_p^2\right) E R_c H_p}{2} - R_p \left(\alpha - \beta R_p - \gamma R_p^2\right) H_e \left(e - \frac{E}{2}\right)
\]

3.1.2. Supplier Point

Setup cost of inventory batch size at the manufacturer is
\[
\frac{M_A}{\omega E} \quad (\therefore \quad Q_m = \omega q_R)
\]

Cost of holding inventory at the manufacturer per annum is
\[
\frac{(\omega - 1) \left(\alpha - \beta R_p - \gamma R_p^2\right) E M_c H_p}{2}
\]

Cost of shipping incurred to the manufacturer is
\[
\frac{M_c}{E}
\]

Annual overall cost of manufacturer is obtained as the sum of cost of ordering per annum, cost of holding and cost of shipping.
\[
M_p(\omega, E) = \frac{M_A}{\omega E} + \frac{M_c}{E} + \frac{(\omega - 1) \left(\alpha - \beta R_p - \gamma R_p^2\right) E M_c H_p}{2}
\]
3.1.3. **Overall Supply Chain**

If the manufacturer and retailer are decided to follow the mutual optimal inventory decision policy, the expression derived for the annual overall cost of supply chain is obtained by adding the terms signifying annual overall cost of the buyer and supplier.

The joint annual overall cost of the retailer and manufacturer, \( S_{\psi} (\omega, E) \) is expressed as

\[
S_{\psi} (\omega, E) = R_{\psi} + M_{\psi}
\]

(4)

**Case I: \( E \geq e \)**

With further generalization and rearrangement of the terms in the equation 4, the annual overall cost of the supply chain is expressed as

\[
S_{\psi} (\omega, E) = \\
\frac{1}{E} \left( R_{A} + R_{z} + \frac{M_{A}}{\omega} + M_{z} \right) + \frac{\alpha - \beta R_P - \gamma R_P^2}{2} \left( R_{C} + (\beta - 1)M_{C} \right) + \frac{\left( \alpha - \beta R_P - \gamma R_P^2 \right)(E-e)^2 R_{c} H_{P}}{2E} - \frac{R_{P} \left( \alpha - \beta R_P - \gamma R_P^2 \right) e^2 H_{c}}{2E}
\]

(5)

The optimal value of \( E \) and \( \omega \) that minimize the annual overall cost of the supply chain follow the optimality conditions shown below.

**Proposition 1:** For the known value of \( \omega \), if the condition

\[
\frac{2}{E^3} \left( R_{A} + R_{z} + \frac{M_{A}}{\omega} + M_{z} \right) > \left( \alpha - \beta R_P - \gamma R_P^2 \right) \left( \frac{e}{E^3} \right) \left( R_{c} H_{P} - R_{p} H_{p} \right)
\]

(6)

is fulfilled, then, the annual overall cost of supply chain, \( S_{\psi} (\omega, E) \) is a convex function of \( E \) and the optimal value of \( E \), \( E^* \) is obtained from the following equation.

\[
E^* = \left( \frac{2 \left( R_{A} + R_{z} + \frac{M_{A}}{\omega} + M_{z} \right) + \left( \alpha - \beta R_P - \gamma R_P^2 \right) e^2 \left( R_{c} H_{P} - R_{p} H_{p} \right)}{\left( \alpha - \beta R_P - \gamma R_P^2 \right) H_{P} \left( 2R_{C} + (\omega - 1)M_{C} \right)} \right)^{0.5}
\]

Proof: By taking the first-order partial derivative of the Equation (5) with respect to \( E \), we have the following equation.

\[
\frac{\partial}{\partial E} \left( S_{\psi} (\omega, E) \right) = \frac{1}{E^2} \left( \frac{2 \left( R_{A} + R_{z} + \frac{M_{A}}{\omega} + M_{z} \right) + \left( \alpha - \beta R_P - \gamma R_P^2 \right) e^2 \left( R_{c} H_{P} - R_{p} H_{p} \right)}{\left( \alpha - \beta R_P - \gamma R_P^2 \right) H_{P} \left( 2R_{C} + (\omega - 1)M_{C} \right)} \right)
\]

(7)

Again, by considering the second-order partial derivative of Equation (5) with respect to \( E \),

\[
\frac{\partial^2}{\partial E^2} \left( S_{\psi} (\omega, E) \right) = \frac{2}{E^3} \left( R_{A} + R_{z} + \frac{M_{A}}{\omega} + M_{z} \right) + \left( \alpha - \beta R_P - \gamma R_P^2 \right) \left( \frac{e}{E^3} \right) \left( R_{c} H_{P} - R_{p} H_{p} \right)
\]

(8)

Now, from the Condition (7) and Equation (8), we have, \( \frac{\partial^2}{\partial E^2} \left( S_{\psi} (\omega, E) \right) > 0 \) i.e. the principal minor of the Hessian matrix is greater than zero. Therefore, the value of annual overall cost of the supply chain, \( S_{\psi} (\omega, E) \) is a convex function of \( E \). Next, for finding the
optimal value of $E, E^*$, the first-order partial derivative of $S_\psi(\omega,E)$ with respect to $E$ is equated to zero. Hence,

$$\frac{\partial}{\partial E}(S_\psi(\omega,E)) = 0$$

This gives that,

$$\left\{2\left(R_A+R_c+\frac{M_\Lambda}{\omega}+M_c\right)+\left(\alpha-\beta R_p-\gamma R_p^2\right)e^2\left(R_cH_p-R_pH_e\right)\right\}$$

$$= E^2\left(\alpha-\beta R_p-\gamma R_p^2\right)H_p\left(2R_c+(\omega-1)M_c\right)$$

Solving the Equation (9) for the cycle time $E$, we obtain the optimal solution of $E, E^*$ as follows.

$$E^* = \left\{\frac{2\left(R_A+R_c+\frac{M_\Lambda}{\omega}+M_c\right)+\left(\alpha-\beta R_p-\gamma R_p^2\right)e^2\left(R_cH_p-R_pH_e\right)}{\left(\alpha-\beta R_p-\gamma R_p^2\right)H_p\left(2R_c+(\omega-1)M_c\right)}\right\}^{0.5}$$

**Proposition 2:** Correspondingly, for the given value of cycle time, $E$, if the inequality condition

$$\omega^*\left(\omega^*-1\right) \leq \frac{2M_\Lambda}{\left(\alpha-\beta R_p-\gamma R_p^2\right)E^2M_cH_p} \leq \omega^*\left(\omega^*+1\right)$$

is satisfied, the expression signifying the annual overall cost of supply chain is convex in terms of number of deliveries, $\omega$.

**Proof:** For given value of $E$, the optimal value of $\omega, \omega^*$ satisfies the next two-inequality conditions.

$$S_\psi\left(\omega^*\right) \leq S_\psi\left(\omega^*-1\right) \text{ and } S_\psi\left(\omega^*\right) \leq S_\psi\left(\omega^*+1\right)$$

By substituting the relevant values in equation (5) for the inequality condition

$$S_\psi\left(\omega^*\right) \leq S_\psi\left(\omega^*-1\right)$$

and with further simplification, the following inequality condition is obtained.

$$\omega^*\left(\omega^*-1\right) \leq \frac{2M_\Lambda}{\left(\alpha-\beta R_p-\gamma R_p^2\right)E^2M_cH_p}$$

Likewise, upon substitution of relevant values in equation (5) for the condition

$$S_\psi\left(\omega^*\right) \leq S_\psi\left(\omega^*+1\right)$$

and with further simplification, the following inequality condition is obtained.

$$\frac{2M_\Lambda}{\left(\alpha-\beta R_p-\gamma R_p^2\right)E^2M_cH_p} \leq \omega^*\left(\omega^*+1\right)$$

By combining equations (12) and (13), the inequality fulfilling optimality for number of deliveries is obtained as
Case II: \( E < e \)

With further generalization and rearranging the terms in the equation 4, the annual overall cost of the supply chain is expressed as

\[
\phi (\beta, T) = \frac{1}{E} \left[ R_A + R_T + \frac{M_A}{\omega} + M_T \right] + \frac{(\alpha - \beta P_R - \gamma P_F^2)^2}{2} \left[ E_H P + (\omega - 1)M_C \right] - R_p \left( \alpha - \beta P_R - \gamma P_F^2 \right) I_e \left( e - \frac{E}{2} \right) \quad (15)
\]

The optimal value of \( E \) and \( \omega \) that minimize the annual overall cost of the supply chain follow the optimality conditions presented below.

**Proposition 3**: For the known value of \( \omega \), if the condition

\[
2 \left( R_A + R_T + \frac{M_A}{\omega} + M_T \right) > 0 \quad (16)
\]

is fulfilled, then, the annual overall cost of supply chain, \( (S_\psi (\omega, E)) \) is a convex function of \( E \) and the optimal value of \( E \), \( E^* \) is obtained from the following equation.

\[
E^* = \left[ \frac{2 \left( R_A + R_T + \frac{M_A}{\omega} + M_T \right)}{(\alpha - \beta P_R - \gamma P_F^2)^2 \left[ E_H P + (\omega - 1)M_C \right] + R_p H_e} \right]^{0.5}
\]

**Proof**: By taking the first-order partial derivative of the Equation (15) with respect to \( E \), we obtain the following equation.

\[
\frac{\partial}{\partial E} (S_\psi (\omega, E)) = \frac{1}{E^2} \left( R_A + R_T + \frac{M_A}{\omega} + M_T \right) + \frac{(\alpha - \beta P_R - \gamma P_F^2)^2}{2} \left[ E_H P + (\omega - 1)M_C \right] - R_p \left( \alpha - \beta P_R - \gamma P_F^2 \right) I_e \left( e - \frac{E}{2} \right) \quad (17)
\]

Again, by considering the second-order partial derivative of Equation (15) with respect to \( E \),

\[
\frac{\partial^2}{\partial E^2} (S_\psi (\omega, E)) = \frac{2}{E^3} \left( R_A + R_T + \frac{M_A}{\omega} + M_T \right) \quad (18)
\]

Now, from the Condition (16) and the Equation (18), we have, \( \frac{\partial^2}{\partial E^2} (S_\psi (\omega, E)) > 0 \) i.e. the principal minor of the Hessian matrix is greater than zero. Therefore, the value of annual overall cost of the supply chain, \( (S_\psi (\omega, E)) \) is a convex function of \( E \).
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Next, for finding the optimal value of $E$, $E^*$, the first-order partial derivative of $S_\psi(\omega,E)$ with respect to $E$ is equated to zero. Hence, $\frac{\partial}{\partial E}(S_\psi(\omega,E)) = 0$. This gives that

$$2\left(R_A + R_z + \frac{M_A}{\omega} + M_z\right) = E^2 \left(\alpha - \beta R_p - \gamma R_p^2\right)\left[H_P \left(R_C + (\omega-1)M_C\right) + R_P H_z\right] \tag{19}$$

Solving Equation (19) for the cycle time $E$, we obtain the optimal solution of $E$, $E^*$.

$$E^* = \left[\frac{2\left(R_A + R_z + \frac{M_A}{\omega} + M_z\right)}{\left(\alpha - \beta R_p - \gamma R_p^2\right)\left[H_P \left(R_C + (\omega-1)M_C\right) + R_P H_z\right]}\right]^{0.5} \tag{20}$$

**Preposition 4:** Correspondingly, for the given value of $E$, if the inequality condition $\omega^*(\omega^* - 1) \leq \frac{2M_A}{\left(\alpha - \beta R_p - \gamma R_p^2\right)E^2M_C H_P} \leq \omega^*(\omega^* + 1)$ is satisfied, the annual overall cost of supply chain is convex in terms of number of deliveries ($\omega$).

**Proof:** For the given value of $E$, the optimal value of $\omega$, $\omega^*$ satisfies the next two-inequality conditions.

$$S_\psi(\omega^*) \leq S_\psi(\omega^* - 1) \quad and \quad S_\psi(\omega^*) \leq S_\psi(\omega^* + 1)$$

By substituting relevant values in equation (15) for the inequality condition $S_\psi(\omega^*) \leq S_\psi(\omega^* - 1)$ & simplifying, the subsequent inequality condition is obtained as

$$\omega^*(\omega^* - 1) \leq \frac{2M_A}{\left(\alpha - \beta R_p - \gamma R_p^2\right)E^2M_C H_P} \tag{21}$$

Similarly, substituting relevant values in equation (15) for the inequality condition $S_\psi(\omega^*) \leq S_\psi(\omega^* + 1)$ and simplifying, the subsequent inequality condition is derived as

$$\frac{2M_A}{\left(\alpha - \beta R_p - \gamma R_p^2\right)E^2M_C H_P} \leq \omega^*(\omega^* + 1) \tag{22}$$

By combining the conditions (21) and (22), the inequality expression fulfilling the optimality for number of deliveries is obtained as

$$\omega^*(\omega^* - 1) \leq \frac{2M_A}{\left(\alpha - \beta R_p - \gamma R_p^2\right)E^2M_C H_P} \leq \omega^*(\omega^* + 1) \tag{23}$$

**Case III: (e = E)**

When the optimal cycle time becomes equal to the credit period, the annual overall cost of the supply chain is expressed as:

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http://www.iaeme.com/ IJMET/index.asp 308  editor@iaeme.com
\[ S_\psi (\omega, E) = \frac{1}{E} \left( R_A + R_t + \frac{a}{\alpha} + M_t \right) + \frac{\left( \alpha - \beta R_p - \gamma R_p^2 \right) EH_p}{2} \left( R_c + (\omega - 1) M_c \right) - \frac{R_p \left( \alpha - \beta R_p - \gamma R_p^2 \right) EH_e}{2} \]  

(24)

**Proposition 5:** For the known value of \( \omega \), if the condition

\[ \frac{2}{E^3} \left( R_A + R_t + \frac{M_A}{\omega} + M_t \right) > 0 \]

is fulfilled, then, the annual overall cost of supply chain, \( (S_\psi (\omega, E)) \) is a convex function of \( E \) and the optimal value of \( E, E^* \) is obtained from the following equation.

\[ E^* = \left( \frac{2 \left( R_A + R_t + \frac{M_A}{\omega} + M_t \right)}{ \left( \alpha - \beta R_p - \gamma R_p^2 \right) H_p (R_c + (\omega - 1) M_c) - \left( R_p \left( \alpha - \beta R_p - \gamma R_p^2 \right) H_e \right) } \right)^{0.5} \]

Proof: By taking the first-order partial derivative of the Equation (24) with respect to \( E \), we have the following equation

\[ \frac{\partial}{\partial E} (S_\psi (\omega, E)) = -\frac{1}{E^2} \left( R_A + R_t + \frac{M_A}{\omega} + M_t \right) + \frac{\left( \alpha - \beta R_p - \gamma R_p^2 \right) H_p (R_c + (\omega - 1) M_c)}{2} \]

\[ - \frac{R_p \left( \alpha - \beta R_p - \gamma R_p^2 \right) H_e}{2} \]

(26)

Again, by carrying second-order partial derivative of Equation (24) with respect to \( E \),

\[ \frac{\partial^2}{\partial E^2} (S_\psi (\omega, E)) = \frac{2}{E^3} \left( R_A + R_t + \frac{M_A}{\omega} + M_t \right) \]

(27)

Now, from the Condition (25) and the Equation (27), we have, \( \frac{\partial^2}{\partial E^2} (S_\psi (\omega, E)) > 0 \) i.e. the principal minor of the Hessian matrix is greater than zero. Therefore, the annual overall cost of the supply chain, \( (S_\psi (\omega, E)) \) is a convex function of \( E \).

Next, for finding the optimal value of \( E, E^* \), the first-order partial derivative of \( S_\psi (\omega, E) \) with respect to \( E \) is equated to zero. Hence, \( \frac{\partial}{\partial E} (S_\psi (\omega, E)) = 0 \). This gives that

\[ \frac{1}{E^2} \left( R_A + R_t + \frac{M_A}{\omega} + M_t \right) = \frac{\left( \alpha - \beta R_p - \gamma R_p^2 \right) H_p (R_c + (\omega - 1) M_c)}{2} - \frac{R_p \left( \alpha - \beta R_p - \gamma R_p^2 \right) H_e}{2} \]

(28)

Solving Equation (28) for cycle time \( E \), we obtain the optimal solution of \( E, E^* \).

\[ E^* = \left( \frac{2 \left( R_A + R_t + \frac{M_A}{\omega} + M_t \right)}{ \left( \alpha - \beta R_p - \gamma R_p^2 \right) H_p (R_c + (\omega - 1) M_c) - \left( R_p \left( \alpha - \beta R_p - \gamma R_p^2 \right) H_e \right) } \right)^{0.5} \]

(29)
Proposition 6: For the given value of $E$, if the inequality condition
\[ \omega^*(\omega^* - 1) \leq \frac{2M_A}{(\alpha - \beta R_p - \gamma R_p^2)E^2M_H} \leq \omega^*(\omega^* + 1) \]
is satisfied, the annual overall cost of supply chain is convex in terms of number of deliveries ($\omega$).

Proof: For the given value of $E$, the optimal value of $\omega$, $\omega^*$ satisfies the next two-inequality conditions.

\[ S_p(\omega^*) \leq S_p(\omega^* - 1) \quad \text{and} \quad S_p(\omega^*) \leq S_p(\omega^* + 1) \]

By substituting relevant values in equation (24) for the inequality condition
\[ S_p(\omega^*) \leq S_p(\omega^* - 1) \]
& simplifying, the subsequent inequality condition is obtained as
\[ \omega^*(\omega^* - 1) \leq \frac{2M_A}{(\alpha - \beta R_p - \gamma R_p^2)E^2M_H} \]

(30)

Similarly, substituting relevant values in equation (24) for the inequality condition
\[ S_p(\omega^*) \leq S_p(\omega^* + 1) \]
and simplifying, the subsequent inequality condition is derived as
\[ \frac{2M_A}{(\alpha - \beta R_p - \gamma R_p^2)E^2M_H} \leq \omega^*(\omega^* + 1) \]

(31)

By combining the conditions (30) and (31), the inequality expression fulfilling the optimality for number of deliveries is obtained as
\[ \omega^*(\omega^* - 1) \leq \frac{2M_A}{(\alpha - \beta R_p - \gamma R_p^2)E^2M_H} \leq \omega^*(\omega^* + 1) \]

(32)

4. Numerical Illustration

In this current section, optimality of inventory decisions and number of deliveries has been calculated for coordinated supply chain model under quadratic price dependent demand. The parameter values of inventory are as follows: $\alpha=9000$; $\beta=5$; $\gamma=0.3$; $R_A=Rs. 300/\text{order}$, $H_p=Rs. 0.15/\text{Re/year}$, $M_A=Rs. 1200/\text{order}$, $M_C=Rs. 100/\text{unit}$, $H_c=Rs. 0.09/\text{Re/year}$, $e=100/360$ years, $P_e=Rs. 160$ per unit, $R_e=Rs. 250$ per shipment, $M_T=Rs. 750$ per shipment, $R_C=Rs. 140$ per unit.

In this model developed, the optimality criterion is derived for cycle time and number of deliveries. MATLAB program is drafted to work out the model. The results representing optimal values of decision variables and objective function are tabulated in Table 1.
Table 1 Variation of the Optimal Values of Decision Variables and Objective Function

<table>
<thead>
<tr>
<th>Description</th>
<th>e=10</th>
<th>e=20</th>
<th>e=30</th>
<th>e=40</th>
<th>e=50</th>
<th>e=60</th>
<th>e=70</th>
<th>e=80</th>
</tr>
</thead>
<tbody>
<tr>
<td>E^* (in years)</td>
<td>0.1316</td>
<td>0.1326</td>
<td>0.1343</td>
<td>0.1366</td>
<td>0.1395</td>
<td>0.2751</td>
<td>0.2751</td>
<td>0.2751</td>
</tr>
<tr>
<td>q^* (in units)</td>
<td>294.8</td>
<td>297.1</td>
<td>300.8</td>
<td>306.0</td>
<td>312.5</td>
<td>616.2</td>
<td>616.2</td>
<td>616.2</td>
</tr>
<tr>
<td>ω^* (integer)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Q^* (in units)</td>
<td>589.6</td>
<td>594.1</td>
<td>601.6</td>
<td>612.0</td>
<td>625.0</td>
<td>616.2</td>
<td>616.2</td>
<td>616.2</td>
</tr>
<tr>
<td>R_v^* (in Rs.)</td>
<td>9866.4</td>
<td>8698.4</td>
<td>7619.6</td>
<td>6625.5</td>
<td>5710.5</td>
<td>7893.1</td>
<td>6997.1</td>
<td>6101.1</td>
</tr>
<tr>
<td>M_v^* (in Rs.)</td>
<td>13988.6</td>
<td>13915.6</td>
<td>13798.0</td>
<td>13641.8</td>
<td>13453.8</td>
<td>8543.2</td>
<td>8543.2</td>
<td>8543.2</td>
</tr>
<tr>
<td>S_v^* (in Rs.)</td>
<td>23855.0</td>
<td>22613.9</td>
<td>21417.6</td>
<td>20267.3</td>
<td>19164.3</td>
<td>16436.3</td>
<td>15540.3</td>
<td>14644.3</td>
</tr>
</tbody>
</table>

Fig. 1 Variation of Decision Variables and Objective Function w.r.t Credit Period

Table 1 and Figure 1 shows that optimal cycle time and buyer’s replenishment quantity initially increases slightly up to certain value of increase in the credit period. With further increase in the value of credit period, the cycle time and buyers replenishment quantity increases sharply and then the variation becomes constant. The variation in number of deliveries remains constant for up to certain value of the increase in the credit period.

With further increase in the value of the credit period, the number of deliveries are sharply reduced and remains constant. On the other hand, the variation in the optimal value of the annual overall cost of the supply chain significantly decreases, with respect to increase in credit period. This is recognized to the actuality that quantity of ordering increases with cycle time increased. As a consequence of this, quantity of orders and number of deliveries are decreased. Therefore, cost of ordering and cost of shipping are decreased significantly, although the cost of carrying is increased slightly.
A Joint Two-Echelon Inventory Model Coupled with Credit Period and Quadratic Price Dependent Demand

Table 2 Optimal Values of Decision Variables and Objective Function

<table>
<thead>
<tr>
<th>Description</th>
<th>Quadratic Price dependent demand</th>
<th>Linear Price dependent demand $\gamma=0$</th>
<th>Constant Price dependent Demand $B = 0, \gamma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^*$ (in years)</td>
<td>0.1395</td>
<td>0.1518</td>
<td>0.1456</td>
</tr>
<tr>
<td>$q^*$ (in units)</td>
<td>312.5</td>
<td>1116.9</td>
<td>1164.4</td>
</tr>
<tr>
<td>$\omega^*$ (an integer)</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Q^*$ (in units)</td>
<td>625.0</td>
<td>1116.9</td>
<td>1164.4</td>
</tr>
<tr>
<td>$R_p^*$ (in Rs.)</td>
<td>5710.5</td>
<td>9332.4</td>
<td>9076.3</td>
</tr>
<tr>
<td>$M_p^*$ (in Rs.)</td>
<td>13453.8</td>
<td>15485.8</td>
<td>16145.0</td>
</tr>
<tr>
<td>$S_p^*$ (in Rs.)</td>
<td>19164.3</td>
<td>24818.1</td>
<td>25221.4</td>
</tr>
</tbody>
</table>

Further from Table 2, it is shown that the annual overall cost of the supply chain is minimum for quadratic price dependent demand when compared with linear and constant demand. The optimal cycle time is reduced in case of quadratic price dependent demand when compared with the linear and constant demand. The replenishment quantity at the buyer is also reduced in case of quadratic price dependent demand when compared with the linear and constant demand. The rate of shipment is more in quadratic price dependent demand case when compared with the linear and constant demand.

Additionally, the sensitivity analysis has been carried out in order to reveal the effect of variation in parameters of the model over the optimal cycle time, buyers’ replenishment rate, number of deliveries from supplier to buyer and the annual overall cost of the supply chain. From Table 3, it is shown that the deviations in parameters of the model influence the optimal decision variables and objective function for the particular values of the credit period. Moreover, from Table 3, for specific values of parameters of the model, it is found that cycle time increases with increased value of the credit period. In addition, it is noticed that the deviation in the behavioural pattern of buyer ordering quantity is same as that of optimal cycle time. Likewise, there is no variation in the optimal value of number of deliveries from supplier to buyer for up to a certain values of the credit period. Further, the number of shipments reduced and then remains constant. However, the annual overall cost of the supply chain significantly reduces with increased value of the credit period.
### Table 3: Sensitivity Analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>in %</th>
<th>e = 10 days</th>
<th>e = 40 days</th>
<th>e = 70 days</th>
<th>e = 100 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td></td>
<td>E' ( q )</td>
<td>E' ( q )</td>
<td>E' ( q )</td>
<td>E' ( q )</td>
</tr>
<tr>
<td>M ( \alpha )</td>
<td>40%</td>
<td>0.2689</td>
<td>0.2713</td>
<td>0.2767</td>
<td>0.2848</td>
</tr>
<tr>
<td></td>
<td>+20%</td>
<td>0.2612</td>
<td>0.2638</td>
<td>0.2693</td>
<td>0.2546</td>
</tr>
<tr>
<td></td>
<td>-20%</td>
<td>0.2452</td>
<td>0.2480</td>
<td>0.2538</td>
<td>0.4955</td>
</tr>
<tr>
<td></td>
<td>-40%</td>
<td>0.3043</td>
<td>0.3073</td>
<td>0.3137</td>
<td>0.3234</td>
</tr>
<tr>
<td>M ( \beta )</td>
<td>40%</td>
<td>0.2612</td>
<td>0.2638</td>
<td>0.2693</td>
<td>0.5355</td>
</tr>
<tr>
<td></td>
<td>+20%</td>
<td>0.2573</td>
<td>0.2599</td>
<td>0.2655</td>
<td>0.5274</td>
</tr>
<tr>
<td></td>
<td>-20%</td>
<td>0.2493</td>
<td>0.2520</td>
<td>0.2578</td>
<td>0.5149</td>
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<tr>
<td></td>
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<td>0.2452</td>
<td>0.2480</td>
<td>0.2538</td>
<td>0.5085</td>
</tr>
<tr>
<td>M ( \gamma )</td>
<td>40%</td>
<td>0.3858</td>
<td>0.3412</td>
<td>0.3470</td>
<td>0.4057</td>
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<tr>
<td></td>
<td>+20%</td>
<td>0.4269</td>
<td>0.2495</td>
<td>0.2501</td>
<td>0.4685</td>
</tr>
<tr>
<td></td>
<td>-20%</td>
<td>0.2603</td>
<td>0.2630</td>
<td>0.2689</td>
<td>0.5212</td>
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<tr>
<td></td>
<td>-40%</td>
<td>0.2336</td>
<td>0.2281</td>
<td>0.2392</td>
<td>0.4685</td>
</tr>
<tr>
<td>P ( \alpha )</td>
<td>40%</td>
<td>0.1922</td>
<td>0.2281</td>
<td>0.2292</td>
<td>0.2902</td>
</tr>
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<tr>
<td></td>
<td>-40%</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>H ( \alpha )</td>
<td>40%</td>
<td>0.2143</td>
<td>0.2193</td>
<td>0.2300</td>
<td>0.4685</td>
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<tr>
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<tr>
<td></td>
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<td></td>
<td>-40%</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>H ( \beta )</td>
<td>40%</td>
<td>0.2532</td>
<td>0.2535</td>
<td>0.2653</td>
<td>0.4685</td>
</tr>
<tr>
<td></td>
<td>+20%</td>
<td>0.2533</td>
<td>0.2548</td>
<td>0.2580</td>
<td>0.5012</td>
</tr>
<tr>
<td></td>
<td>-20%</td>
<td>0.2534</td>
<td>0.2572</td>
<td>0.2653</td>
<td>0.5438</td>
</tr>
<tr>
<td></td>
<td>-40%</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>40%</td>
<td>0.1009</td>
<td>0.2532</td>
<td>0.2866</td>
<td>0.2843</td>
</tr>
<tr>
<td></td>
<td>+20%</td>
<td>0.1200</td>
<td>0.2533</td>
<td>0.2653</td>
<td>0.1980</td>
</tr>
<tr>
<td></td>
<td>-20%</td>
<td>-</td>
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<td>-</td>
<td>-</td>
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<td></td>
<td>-40%</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>( \beta )</td>
<td>40%</td>
<td>0.3044</td>
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</tr>
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<td></td>
<td>+20%</td>
<td>0.2216</td>
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<td>0.3099</td>
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<td>-20%</td>
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<td>0.4010</td>
</tr>
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<td>-40%</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>40%</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
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<td></td>
<td>+20%</td>
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<td>-40%</td>
<td>0.0968</td>
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<td>0.2099</td>
<td>-</td>
</tr>
</tbody>
</table>

**5. CONCLUSIONS**

In the current research work, a mathematical model is formulated for a two-echelon inventory system in which a single kind of product is supplied from a single-seller to a single-buyer, where the supplier provides credit period to the buyer. In development of the present model, the demand is expressed in terms of quadratic function of buyer’s unit selling price. The optimality is attained for the decision variables and objective function in the form of consequence and validation. From the optimality criterion, a computer programme is drafted in MATLAB and the model is worked out. From research findings, it is concluded that the annual overall cost of supply chain is convex with respect to number of shipments from...
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supplier to buyer and optimal cycle time. Also, from the sensitivity analysis, it is observed that all the model parameters have the significant variation over the optimality of buyer’s replenishment quantity and total supply chain cost. The total supply chain cost decreases with the increased value of trade credit.

From the help of this present model managerial decisions like cycle time, Buyer’s replenishment quantity, number of shipment and the total supply chain cost can be decided. In extension to the above the effect of variation in ordering cost/set up cost, buyer unit cost/supplier unit cost, buyer selling price with respect to annual overall cost of supply chain is also observed. Novelty of the present mathematical model and research conclusions can be helpful in consumer supplies industries. The present model can be extended further to a three-echelon model.

REFERENCES


