DEVELOPMENT OF AN INFORMATION ANALYSIS SYSTEM FOR ANALYZING WAVE PROCESSES IN A HOMOGENEOUS MEDIUM

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ABSTRACT

The economic aspect of the efficient use of materials is one of the most important aspects in the design development and construction. The calculation of dynamic loads allows reducing costs and risks that may arise directly in the process. As a result of dynamic loads, elastic waves occur in the body, the exact calculation of which allows assessing the strength and reliability of the entire structure and technology.

The need to analyze wave processes in a deformed body and predict the regularities of its behavior led to the improvement of mathematical methods of problem solution. Due to the development of information technology and easy access to the computing capacity of computers, numerical solution methods have been revised and re-approached, such as: the finite difference method, the splitting method, the method of spatial characteristics, etc.

This article describes the solution of the Lamé problem by the method of bicharacteristics including the splitting method's principles. This optimization of the method of bicharacteristics by means of the splitting method allows making the method of bicharacteristics more convenient and stable in the numerical solution. Based on the optimized method of bicharacteristics, the WavePRG software suite was developed, which allowed comparing the analytical and numerical solution with determining the accuracy and stability of the numerical method. This software allowed obtaining graphs of normal stress at various time points.

Keywords: Method of Bicharacteristics, Lamé Problem, Defining Equations, Characteristic Cones, Solving Difference Equations, Scatter Plot, Stress Tensor, Accuracy, Stability.
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1. INTRODUCTION
The development of many areas of engineering and construction involves the use of current principles of the contact elasticity theory. This theory is frequently applied in calculations of the stress-strain state of machine parts, elements of composite structures in construction, machine and aerospace engineering.

The boundary between the theory and the practice becomes so subtle that it leads to the need to build and transform mathematical models of the contact elasticity theory that could be used in computation and, most importantly, reflect the physical essence of the problem. Its development is stimulated, first of all, by the demands of the machine-engineering, mining, and processing industries. However, many problems of the elasticity theory cannot be solved analytically. For this reason, numerical solution methods have gained a new boost of development.

The large quantity of numerical methods indicates that their various types can be used to solve different problems. At the same time, the calculation accuracy and stability for solving dynamic problems is a priority condition when choosing a numerical method. Many authors have dealt with numerical methods for solving dynamic problems in elastic media in their works. In particular, the method of spatial characteristics was studied by R.J. Clifton [1], G.T. Tarabrin [2, 3], V.N. Kukudzhanov [4, 5], V.V. Reker [6], etc.; the finite elements method was applied in the joint work of Sh.M. Aitaliev, Zh.K. Masanov, N.M. Makhmetov, etc. [7]; the method of boundary integral equations was used in the work of V.G. Bazhenov [8], as well as Sh.M. Aitaliyev, L.A. Alekseeva, Sh.A. Dildabaeva, N.B. Zhanbyrbaeva [9], etc.

In this article, we optimize the bicharacteristics method by adding the splitting method. Thus, the optimized method is convenient and simpler for solving problems of wave propagation and programming its numerical solution.

1.1. Problem formulation
We have considered the Lamé problem of the propagation of infinite waves in case of an impact with a short triangular pulse over time along the axis of symmetry of the elastic half-plane (Fig. 1).

Figure 1. The shape of the external load pulse B.
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At the initial time point \( t \leq 0 \) the body is at rest:

\[
\nu_\alpha = \sigma_{\alpha j} = 0, \alpha, j = 1, 2
\]  

(1)

The external load pulse is represented as:

\[
\sigma_{11}(t) = P_0 \left( 1 - \frac{x}{x_k} \right) \frac{t}{t_m}, \text{for } 0 \leq x \leq x_k \text{ and } 0 \leq t \leq t_m; \]  

(2)

\[
\sigma_{11}(t) = P_0 \left( 1 - \frac{x}{x_k} \right) \frac{t - t_m}{t_k - t_m}, \text{for } 0 \leq x \leq x_k \text{ and } t_k \leq t \leq t_m; \]  

(3)

\[
\sigma_{11}(t) = 0, \text{ for } x \geq 0, t \geq t_k. \]  

(4)

Here, \( P_0 = 1 \), and \( x \) is the variable by coordinate \( x_2; x_k = 2; t_m = 2; t_k = 4 \). The domain geometry and the boundary conditions are shown in Fig. 2 with dimensionless variables as applied to a homogeneous medium.

Figure 2. The domain geometry and the boundary conditions

The task is to study the stress-strain state of an elastic body \( D \) at \( t > 0 \).

2. MATERIALS & EXPERIMENTAL PROCEDURES

2.1. Method of bicharacteristics

To solve the problem with given initial and boundary conditions, we use a dimensionless system of equations, consisting of motion equations and the ratios of the generalized Hooke's law:

\[
\begin{align*}
\nu_1 &= \sigma_{11,1} + \sigma_{22,2} \vspace{5pt} \\
\nu_2 &= \sigma_{21,1} + \sigma_{22,2} \\
\sigma_{11} &= \nu_{1,1} + \gamma_{11}\nu_{2,2} \\
\sigma_{11} &= \gamma_{11}\nu_{1,1} + \nu_{2,2} \\
\sigma_{12} &= \gamma_{12}^2(\nu_{1,2} + \nu_{2,1})
\end{align*}
\]  

(5)

where \( (\nu)_\alpha, \sigma_{\alpha j} \) are components of the velocity and stress tensor; the lower indices after the decimal point denote the derivatives with respect to the corresponding spatial coordinate; the points above denote the derivatives with respect to time.
For convenience, we introduced independent dimensionless variables and the sought-for values[10]:

\[ c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}; c_2 = \sqrt{\frac{\mu}{\rho}} \tag{6} \]

The properties of linearly elastic homogeneous materials are set by density \( \rho \), and Lamé parameters \( \lambda, \mu \).

System (5) is a system of hyperbolic type, in which the characteristic surfaces in 3D space \((x_1, x_2, t)\) represent a hypercone with axes parallel to the time axis (Fig. 3).

\[ \text{Figure 3. Characteristic cones} \]

The generators of these cones coincide with the bicharacteristics of equations (3). To compose an equation for these bicharacteristics and obtain conditions for them, we split the 2D system (3) into 1D ones. This can be done if we sequentially fix one of the spatial variables in system (3) [11]. This technique corresponds to the idea of K.A. Bagrinovsky and S.K. Godunov on the splitting of multidimensional t-hyperbolic systems into 1D systems.

For \( x_k = \text{const} \), we have [12]:

\[
\begin{align*}
\dot{\nu}_{ij} - \sigma_{ij,j} &= a_{ij} \\
\sigma_{ij,j} - \lambda_{ij}\nu_{i,j} &= b_{ij}
\end{align*}
\tag{7}
\]

where \( \alpha_{ij} = \sigma_{ik,k}; \lambda_{ij} = \gamma_{12}(1-\delta_{ij}); b = (\gamma_{12}\delta_{ij} + \gamma_{12}^2(1-\delta_{ij}))\nu_{p,k} \).

Hereinafter, \( i, j, k, p = 1, 2; p \neq i; k \neq j \).

Applying the known methods for deriving the differential equations of bicharacteristics and the conditions for them, we find:

\[ dx_j = \pm \lambda_{ij}dt \tag{8} \]

\[ d\sigma_{ij} \pm \lambda_{ij}d\nu_i = (b_{ij} \pm \lambda_{ij}a_{ij})dt \tag{9} \]

here \( d\nu_i \) is the increment of displacement velocity \( v_i \) along bicharacteristic (8) for time \( dt \).

In equations (8) and (9), the upper sign corresponds to the positively directed bicharacteristics and the lower sign corresponds to the negatively directed bicharacteristics.
2.2. Choice of the scatter plot and template

A study of the characteristic surfaces is required in order to carry out numerical calculations of the problem under consideration with the given configuration (body D). Body D is subjected to dynamic loads in the form of a triangular pulse.

Let the body be divided into square cells with sides \( x_1 = x_2 = h \). At the nodal points, we seek the values of functions \( V_i, \sigma_{ij} \) at different time points with increment \( \tau \). In addition to the mentioned nodal points, the point grid, based on which the difference scheme is drawn, contains points formed by the intersection of bicharacteristics with hyperplanes \( t = \text{const} \) [13].

The template consisting of node O and points \( E_{ij}^{\pm} \), separated from point O by distances \( \lambda_{ij} \tau \) (Fig. 4).

\[ \text{Figure 4. Scatter plot of the difference equations} \]

From this point on, the values of the functions are assigned the upper sign “0” at point O, the subscript \( ij \) and the upper sign “±” at points \( E_{ij}^{\pm} \) (e.g. \( \sigma_{ij}^{\pm} \)), and no additional index is assigned at point A (at the upper layer of time \( t_0 + \tau \)).

Based on the described scatter plots, the method for solving dynamic problems developed below enables us to determine velocities \( V_i \) and the components of the stress tensor \( \sigma_{ij} \) at point A at a certain time layer \( t = t_0 + \tau \), if their values at the previous layer \( t = t_0 \) at point O and adjacent points \( E_{ij}^{\pm} \). Difference schemes of this type are called explicit. These systems are solved successively from one time layer to the next.

2.3. Solving difference equations

Integration of equation (3) from point O to point A and ratios (6) from point \( E_{ij}^{\pm} \) to point A by the method of trapezoids allows obtaining the following expressions:

\[ V_i = v_i^0 + \frac{\tau}{2}(\sigma_{ij,.} + a_{ij} + v_i^0) \]  \( (10) \)

\[ \sigma_{ij} = \sigma_{ij}^0 + \frac{\tau}{2}(\lambda_{ij}^0 V_i + a_{ij} + \sigma_{ij}^0) \]  \( (11) \)

\[ \sigma_{ij} - \sigma_{ij}^{\pm} = \lambda_{ij}(v_i + v_i^\pm) = \frac{\tau}{2}(b_{ij} + b_{ij}^\pm + \lambda_{ij}^0(a + a_{ij}^\pm)) \]  \( (12) \)
Eliminating $\sigma_{ij}$, $V_i$ from (12) with (11), we can obtain:

$$\lambda^2 V_{ij, i, j} + \lambda_{ij} \sigma_{ij, j} = b_{ij}^0 + \sigma_{ij}^0 + \lambda_{ij} (v_i^0 - a_{ij}^0) + \frac{2}{\tau} (\sigma_{ij}^0 - \sigma_{ij}^0 + \lambda_{ij} [v_i^0 - v_i^\pm])$$

(13)

The values of the unknowns at non-node points of expression (12) are found by the Taylor formula near node point $O$ to second-order accuracy with respect to step $\tau$. Thus, the following is obtained:

$$\lambda^2_v V_{i, j} + \lambda_{ij} \sigma_{ij, j} = \lambda^2_v (v_{i, j}^0 + \tau v_{i, j}^0) + \lambda_{ij} (\sigma_{ij, j}^0 + \tau \sigma_{ij, j}^0)$$

(14)

Adding and subtracting each equation of system (14) with identical pairs of indices, we can find:

$$v_{ij} = v_{ij}^0 + \tau (\sigma_{ij, j}^0 + a_{ij}^0)$$

$$\sigma_{ij, j} = \sigma_{ij, j}^0 + \tau (\lambda^2_v v_{ij, j}^0 + b_{ij}^0)$$

(15)

The procedures for obtaining solving systems of equations at the node points of the body under study at time point $t_0 + \tau$ are different for the interior, boundary, and corner points of the domain under study.

For interior points of the domain. Unknown derivatives $\sigma_{ij, j}$, $v_{i, j}$, $a_{ij}$, $b_{ij}$ at layer $t = t_0 + \tau$ are sought for from the system of equations (14). The derivatives of functions on the right-hand side of the system of equations (5) and (15) for the square grid for node $(x_1, x_2, t_0)$ are approximated by central second-order differences.

For boundary points of the domain. Difference equations for the boundary points of the studied domain of plane $t = t_0 + \tau$ (excluding the angular ones) from the calculated or given values of the sought-for values at layer of time $t_0$ are obtained through the system of equations (11) and (14). The calculations cannot use conditions (14) for two characteristics (points $E_{ij}^\pm$) that do not belong to the body domain. Thus, in comparison with interior points, the number of equations (14) is reduced by two. The set of remaining equations (11), (14) and the two boundary conditions is a closed linear system with respect to thirteen unknowns. To approximate the derivatives of functions, the forward and backward differences are used.

2.4. Accuracy and stability of the numerical solution

The system of difference equations (5) should lead to a solution that coincides with the solution of the original system. The general theory of partial differential equations for this requires certain restrictions to the grid ratio of steps with respect to time and coordinate in problems with initial and boundary conditions that can be represented as follows [14]:

$$\left| \frac{\tau c_{aj}}{h} \right| < 1$$

(16)

where $c_{aj}$ are the coefficients of the hyperbolic system. Physically, such a restriction means that the solution at the hypercone vertex is expressed in terms of the initial value inside the domain bounded by the hypercone surface, i.e. the solution at the sought-for point is determined through the influence domain. If such a restriction is not met, the solution at points near the boundary will depend on the initial data outside the domain under
consideration, and in this case, the convergence of the solutions of the difference equations to the solutions of the differential equations cannot be expected.

At any point in the domain under consideration, there is a method error that grows as the calculation time increases. The effect of such errors at neighboring points on the solution result at the sought-for point becomes lower as the grid ratio decreases. In expression (15), unknown functions and unknown derivatives at the node points of the domain under study are in some cases calculated by quadratic and linear interpolation according to the Lagrange formula. The interpolation is accompanied by the disturbance propagation acceleration effect. Obviously, this effect occurs not only on the boundary wave, but also in the presence of a wave interference process and, therefore, can greatly distort the wave-interference pattern. In this connection, the difference scheme based on the interpolation with respect to nodal points in the direction of the axes of spatial variables, can be recommended only in those cases when the wave interference is accounted for purposefully or this process is weakly expressed. The distortions introduced by the interpolation procedure are the smaller, the smaller the grid step along the spatial variables is [15].

A necessary condition for the stability of the grid-characteristic method, which follows from the Neumann condition (the spectral radius of the augmented matrix does not exceed one), is found as follows:

\[
\max \left\{ \tau \lambda_{ii}, \frac{\tau \lambda_{ij}}{h} \right\} \leq 1
\]  

which expresses the Courant-Friedrichs-Lewy condition. During further calculations, the steps of the spatiotemporal grid are selected according to the stability conditions (16) and (17). Numerous calculations have experimentally verified that condition \( \tau \leq 1 \) ensures the stability of counting for a large time point [16].

3. RESULTS

To solve this problem, we developed the WavePRG software. WavePRG is based on an optimized method of bicharacteristics and uses the splitting method. The solution algorithm is implemented in the algorithmic language in a rectangle with length \( x2 = \frac{4}{3} \), width \( x1 = 1 \), and the grid quadrature step \( h = \frac{1}{45} \). Poisson's ratio \( \nu = 0.3 \).

Due to the symmetry of the applied load, parameters \( \sigma_11, \sigma_22, \nu_1 \) are the even, \( \sigma_12 \) and \( \nu_2 \) are the odd functions with respect to axis \( x2 = 0 \). The solution is provided for the domain \( x1 \geq 0 \). Boundary \( x1 = 0 \) is stress-free:

\[
\sigma_{11} = 0; \quad \sigma_{12} = 0
\]  

everywhere except at points where impulse loads are applied (Fig. 1, 2). The radiation conditions were set at boundaries \( x1 = x0 \) and \( x2 = x0 \). According to the latter, assume that, for example, for \( x1 = x0 \) the motion is close to one-dimensional and perpendicular to the boundary. Thus, at boundary \( x2 = x0 \), the following conditions are set:

\[
\sigma_{22} + \lambda_{22} \nu_2 = 0; \quad \sigma_{12} + \lambda_{12} \nu_1 = 0,
\]
and at boundary $x_1 = x_0^1$, the following conditions are set:

$$\sigma_{11} + \lambda_{11} \nu_1 = 0; \sigma_{21} + \lambda_{21} \nu_1 = 0,$$

The problem solution reduces to integrating equation (5) for the initial (1) and boundary (3), (18)–(20) conditions and can be found by the difference method for a plane stressed state of the homogeneous medium under study.

The difference solution (Fig. 5) found with the software allows comparing with the result of the analytical solution (Fig. 6) at the point with coordinates $x_1 = 0, x_2 = \frac{2}{3}$ depending on time $t$ and shows a slight error.

This, in turn, proves the high accuracy of the numerical solution. The obtained accuracy of the solution by this method can be considered satisfactory for engineering purposes; moreover, the qualitative nature of the behavior of the sought-for values is correctly conveyed.

**Figure 5.** Change in normal stress $\sigma_{22}$ over time at point $x_1 = 0, x_2 = \frac{2}{3}$ for the difference solution of the Lamé problem.

**Figure 6.** Change in normal stress $\sigma_{22}$ over time at point $x_1 = 0, x_2 = \frac{2}{3}$ (dashed lines are the exact solution, solid lines are the difference solution of the Lamé problem).

In this article, we use the WavePRG software to verify the solution accuracy and finds diagrams of the change in the normal stress at time points $60\tau$ (Fig. 7) and $120\tau$ (Fig. 8).
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Figure 7. Change in normal stress $\sigma_{22}$ by $x_2/h$ for $x_1 = 0$ at time point $t = 60\tau$.

Figure 8. Change in normal stress $22$ by $x_2/h$ for $x_1=0$ at time point $t = 120\tau$.

Thus, the developed calculation method can be used to analyze the stress-strain state and the features of the dynamic perturbation propagation in homogeneous plane bodies under the action of longitudinal, transverse, and impulse loads, and the WavePRG software allows automating the calculation process.

4. CONCLUSION

It seems hardly feasible to accurately solve a variety of wave propagation problems by the analytical method. For this reason, the use of numerical methods is a good alternative and is applied in the design development and construction of engineering facilities and structures in order to use structural materials more efficiently.

The ability to predict the dynamic load can invaluably contribute to reducing or possibly eliminating risks in construction.

We have proposed the solution of the Lamé problem by the method of bicharacteristics. The developed numerical method has demonstrated its high accuracy and stability, which, in turn, suggests a wide range of applications of this method and the possibility to use it for solving wave problems. The advantage of this method is that it allows approaching the domain of maximum dependence of the finite and differential equations to the dependence domain of the initial differential equation.

The analysis, stability check, and accuracy determination of the numerical solution were provided with the WavePRG software.

The obtained results and software can be used in wave process studies in construction, engineering, machine-building, etc.
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