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OPTIMAL STOCHASTIC CONTROL PRINCIPLE AND ITS' APPLICATION IN FACTORY CONSUMPTION MODEL

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ABSTRACT

This paper deal with the optimal stochastic control principle and its' application in formulating consumption model of a production company a case study of Landmark University development ventures (LMDV). Here Stochastic Differential Equations (SDE) is considered as an ordinary differential equations (ODE) driven by white noise and we justified the connection between the Ito's integral and white noise in the case of non-random integrands interpreted as cost functions.

Keywords: Investment, Optimal, Stochastic, Venture, White Noise,Production.

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1. INTRODUCTION

Optimal Stochastic principle via dynamic models and control system played vital roles in solving various problems in engineering, management and economics. It is used to examined and calculate the conditions of operation for an optimal industrial process to minimized the production cost and maximized both profits and efficiency. At the beginning of every fiscal year, each production firms do render their budgetary which normally includes both investment and production planning and their expected profits. Sethi and Lee (1981) applied the calculus of variation principle to solve a production investment model. Merton (1969) provides a nice introduction in the applications of optimal control theory to investment and production models using continuous time case. Merton (1971) Derzko and Sethi (1981) introduces dynamics of economics and management models using stochastic version. Oladejo et.al (2019) established optimal principle in solving over-allocation and under-allocation of the classroom space using Linear Programming based on the data obtained from the examination and lecture timetable committee on the classroom facilities, capacities and the number of students per programme to maximize the available classroom space and minimizes the congestion and overcrowding in a particular lecture room using AMPL software Likewise Oladejo et.al (2019) Applied optimization principle in optimizing profits of a production industry using linear programming where they examine and evaluate production costs to determine the optimal profit using secondary data collected from the records of the Landmark University Bakery on five types of bread produced in the firm where it was revealed through the application of AMPL software that Family loaf and the Chocolate bread contributed objectively to the profit. Hence, more of Family loaf and Chocolate bread are needed to be produced and sold in order to maximize the profit.

Dockner and Feichtinger (1993) opined that optimal inter-temporal price and production decisions depending on the sum of the adjoint variable of the inventory level and the Lagrange multiplier of the state constraint by using the optimal control theory. El-Gohary (2006) considered optimal investment problem for the Nerlove-Arrow model under a replenish-able budget where an optimal control problem with two state variables for the dynamics of this model and the optimal control is the rate of investment expenditure that is required to maximize the present value of net streams over an infinite time horizon subject to a replenish-able budget

2. STOCHASTIC DIFFERENTIAL EQUATIONS

We let our controlled stochastic differential equation (SDE) be of the form:

$$\dot{X}(t) = f\left(t, x(t)\right)dt + \sigma\left(t, (x(t)\right)dw(t) \tag{1}$$

With initial condition $x(0)$, this leads to integral equation of the form:

$$x(t) = x(0) + \int_0^t f\left(s, x(s)\right)ds + \int_0^t \sigma\left(s, x(s)\right)dw(s). \tag{2}$$

The solution $x(t)$ of the equation (2) gives the differential equation of the form:

$$\dot{X}(t) = \bar{f}(t)dt + \bar{\sigma}(t)dw(t) \tag{3}$$

Where

$$\bar{f} = f\left(t, x(t)\right) \text{ and } \bar{\sigma}(t) = \sigma\left(t, x(t)\right) \tag{4}$$

But,

$$\int_0^t \bar{f}(s)ds < \infty \text{, and } \int_0^t \bar{\sigma}^2(s)ds < \infty \tag{5}$$

It is remarkable to note that equation (1) can be written as

$$dx(t) = f\left(t, x\right) + \sigma\left(t, x\right)\psi \tag{6}$$

Where ψ represent the white noise.

Equation (1) can be interpreted as a scalar equation or as a vector equation.

x, f represent $n-$vectors, w represent $r-$vector

Where r, is the Wiener process serves as component and σ represent $n \times r$ matrix.

By linearity, the scalar equation (1) can be read as:

$$\dot{X} = (\alpha + \beta x)dt + (\lambda + \delta x)dw. \tag{7}$$

Where $\alpha, \beta, \lambda,$ and δ are constants or time-dependent scalar.
Equation (7) can be replaced in the linear case by

$$\dot{X} = (a + Ax)dt + \sum_{i=1}^{r}(b_i' + B_i x)dw_i \tag{8}$$

Where a, b_i are vectors and A, B_i represent matrices

2.1. Stochastic Optimal Control Principle/Theory

For any given system represented by a differential equation of the form;

$$dx = f(t, x_t, u_t, z_t) = f(x_t, u_t, t) + Gdz(t). \tag{9}$$

An optimal control problem is specified by giving a performance criterion that grades the possible control function u in order of preference by attaching a number $J(u)$ to anyone.

$J(u)$ is refers to and called the cost of u ,so that we can choose the control that minimizes it. If $J(u)$ represents the profit then we can maximize it by minimizing $-Ju$, meanwhile the distinction between the minimizing and maximizing is purely notational.

In the optimal control theory, the type of cost function considered is almost invariably written as:

$$J(u) = E\left\{\int_0^T H(t, x_t, u_t)dt + G(x_t)\right\} \tag{10}$$

Where T is infinite termination time and not fixed.

In investments problems where x_t is the value of one's asset. H represents the consumption rate, but in engineering perspective, H is usually chosen to be the cost deviation from some desired trajectory of x_t or the use of too much control forces or energy. $G(x_t)$ is the cost failure to reach some special target set at terminal.

Many deterministic optimal control problems can be formulated in other to have a cost

$$\int_0^T H(t, x_t, u_t, z_t)dt + G(x_t). \tag{11}$$

Solving the stochastic optimal control problems defined in the equations (10) and (11)
We let $V(x,t)$ refers to as the current value function be the expected value of the objective function of the equation (11) from t to T:

$$V(x,t) = \max_u E\left\{\int_t^T H(x, u, z, t)dt + G(x_t)\right\} \tag{12}$$

When an optimal policy is followed from t to T ,

Given $X_t = x.$ then by the optimality principle,

$$V(x,t) = \max_u E\left[H(x, u, z, t)dt + V(x + dx_t, t + dt)\right]. \tag{13}$$

Applying Taylor's expansion we get;

$$V(x+dx_t, t+dt) = V(x,t) + V_t dt + V_x dx_t + \frac{1}{2}V_{xx}(dx_t)^2 + \frac{1}{2}V_{tt}(dt)^2 + \frac{1}{2}V_{xt}dx_t dt \quad (14)$$

Using the equation (10) we write formally thus:

$$\left(\dot{X}(t)\right)^2 = f^2(dt)^2 + G^2(dz_t)^2 + 2fGdz_t dt. \quad (15)$$

$$dx - tdt = f(dt)^2 + Gdz_t dt.. \quad (16)$$

Here, it is sufficient to know the multiplication rules of the stochastic calculus

Thus: $(dz_t)^2 = dt, dz_t dt = 0, (dt)^2 = 0.$ (17)

Substitute from the equation (14) into the equation (13), apply the equations (10), (11) and equation (12) we get:

$$V(x,t) = \max_u\left[Hdt + V(x,t) + V_t dt + V_x fdX_t + \frac{1}{2}V_{xx}G^2 dt + 0(dt)\right]. \quad (18)$$

Conceding the term $V(x,t)$ on both sides of the equation (18), dividing the remainder by dt

Given that $dt \to 0$, then following the Hamilton-Jacob-Bellman equation can be derived

$$0 = \max_u\left[H + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f + \frac{1}{2}G^2\frac{\partial^2 V}{\partial x^2}\right]. \quad (19)$$

For the current value function $V(x,t)$, with the boundary condition

$$V(x,T) = S(x,T) \quad (20)$$

3. APPLICATION OF OPTIMAL STOCHASTIC TO CONSUMPTION MODEL

Here we formulate the deterministic consumption model. We assume that $w(t)$ denotes the wealth at time t, which represents the state variable, and $C(t)$ is the rate of consumption, which represents the control variable at time t.

From the differential equation of the form:

$$dw(t) = rw(t) - C(t), w(0) = w_0 \quad (21)$$

Where r is the compounded rate.

We formulate the optimal control problem of the form:

$$\max\left\{J = \int_0^T e^{-\rho t} In[C(t)]dt + e^{-\rho T}BW(T)\right\} \quad (22)$$

Subject to:

$$dw = rw - C, w(0) = w_0, w(T) \geq 0 \quad (23)$$

Where T is the time, B is constant, $W(T)$ denote the wealth.

We let R_0 be the initial price of an investment, r the interest rate and R_t be an accumulated changing rate at time t

Then,

$$dR_t = \frac{dR_t}{dt} = rR_t, R_0 = R(0) \tag{24}$$

By applying separation of variable method, the equation yields;

$$dR_t = rR_t dt, \qquad R_0 = R(0) \tag{25}$$

Solving, we get the accumulated amount as a fraction of time;

$$\text{Thus;} \quad R_t = R_0 e^{rt}$$

[11] and [12] suggested that the stock price p_t can be formulated by Ito's Stochastic differential equation of the form:

$$dp_t = \alpha p_t dt + \sigma p_t dz_t \qquad p_0 = p(0) \tag{26}$$

Where α represents the expected value of the return rate on stock, σ^2 is the variance associated with the return and z_t represents a standard wiener process.

3.1. Parameters use in Stochastic Optimal Model

We considered the following parameters to formulate the stochastic optimal control model.

$Y_t = x =$ The wealth at time t

$K_t =$ The consumption rate at time t

$Z_t =$ The fraction of the wealth invested in stock at time t

$1 - Z_t =$ The fraction of the wealth in saving account at time t

$V_c =$ The utility of consumption at consumption rate c

$\rho =$ The discount rate applied to consumption utility.

$B =$ The bankruptcy parameter

We develop the dynamics of the consumption model and consider the wealth equation of the form:

$$dY_t = [(\alpha - r)Z_t Y_t + (rY_t - K_t)]dt + \sigma Z_t Y_t dz_t \quad Y_0 = Y(0) \tag{27}$$

Where

$\alpha Z_t Y_t dt$ is the expected return from the risky investment $Z_t Y_t$ at time t to $t + dt$.

$\sigma Z_t Y_t dt$ is the risk involved in investing $Z_t Y_t$ in the stock.

$r(1 - Z_t)Y_t dt$ is the amount of interest earned on the balance of $(1 - Z_t)Y_t$.

$K_t dt$ is the amount of consumption during the time interval from t to $t + dt$.

Equation (27) show that one can trade continuously in time without incurring any broker's commission and the changes dY_t in wealth from t to $t + dt$ is due to the gain from changes in share and consumption.

We then formulate the problem of optimal control of stochastic consumption model as.

$$\max_{K_t>0}\left\{J=E\left[\int_0^T e^{-\rho t}V(K_t)dt+Be^{-\rho T}\right]\right\} \tag{28}$$

Subject to;

$$dY_t=\left[(\alpha-r)Z_tY_t+(rY_t-K_t)\right]dt+\sigma Z_tY_tdz_t,\ Y_0=Y(0),\ K_t>0 \tag{29}$$

The bankruptcy parameter denoted by B can be positive if there is a social welfare system in place and can also be negative if there is a remorse associated with bankruptcy.

We let $V(x)$ be the value function associated with an optimal policy with $Y_t=x$ at time t.

Then, the Hamilton-Jacobi-Bellman equation satisfied by the value function $V(x)$ is of the form:

$$\rho V(x)=\max_{K_t\geq 0}\left\{(\alpha-r)qx\frac{dV}{dx}+(rx-C)\frac{dV}{dx}+\frac{1}{2}q^2\sigma^2x^2\frac{d^2V}{dx^2}+V(C)\right\},V(0)=B. \tag{30}$$

$$\text{We let } V(C)=\sqrt{C} \tag{31}$$

$$\frac{dV}{dC}\bigg|C=0=\frac{1}{2\sqrt{C}}\bigg|C=0=\infty \tag{32}$$

Differentiating equation (30) with respect to Q and C, equating it to zero yield:

$$\left.\begin{array}{l}(\alpha-r)x\dfrac{dV}{dx}+Z\sigma^2x^2\dfrac{d^2V}{dx^2}=0\\ 1-c\dfrac{dV}{dx}=0\end{array}\right\} \tag{33}$$

Solving equation (33) with respect to the function $Z(x)$ and $K(x)$

$$Z(x)=-\left(\frac{r-\alpha}{\sigma^2x}\right)\left(\frac{dV}{dx}\right)\left(\frac{d^2V}{dx^2}\right)^{-1} \tag{34}$$

And

$$K(x)=\left(\frac{dV}{dx}\right)^{-1} \tag{35}$$

Substituting equations (35) and (34) into the equation (30) yields;

$$\left[rx\frac{dV}{dx}-In\left(\frac{dV}{dx}\right)\right]\frac{d^2V}{dx^2}-\lambda\left(\frac{dV}{dx}\right)^2-\rho V=0 \tag{36}$$

Where

$$\lambda=\frac{(\alpha-r)^2}{2\sigma^2}$$

4. SOLUTION TO STOCHASTIC CONSUMPTION MODEL

The nonlinear ordinary differential equation (18) was solved and it takes the form:

$$V(x)=aIn(bx)+D \tag{37}$$

Where a, D are constant and determined by:

$$\frac{dV}{dx} = \frac{1}{ax}, \frac{d^2V}{dx^2} = -\frac{1}{ax^2} \tag{38}$$

Find the constants a, b, D by substituting equations (38) and (37) into the equation (36). Thus;

$$a = \frac{1}{\rho}, b = \rho, D = \frac{r - \rho + \lambda}{\rho^2} \tag{39}$$

And the solution to the equation (65) is given as:

$$V(x) = \frac{1}{\rho} In\rho x + \frac{r - \rho + \lambda}{\rho^2} \tag{40}$$

Substituting equation (40) into the equation (34) by calculation, the wealth invested in the stock is;

$$Z = \frac{\alpha - r}{\sigma^2} \tag{41}$$

Then, the optimal consumption rate Vc is;

$$V = \rho x \tag{42}$$

5. NUMERICAL EXAMPLES

Here we find the optimal consumption rate and the fraction of the wealth invested in the stock

Example 1 Given that the utility against consumption rate $Vc = \sqrt{V}$ .Table1 below shows the expected current values against rate level of the utility with consumption rate and Fig 1 shows the expected optimal inventory level against time

Table 1

R	$V_a[x]$	$V_b[x]$	$V_c[x]$
0	0.08267	0.30145	0.080262
10	0.08367	0.31745	0.080956
20	0.08467	0.33345	0.081656
30	0.08567	0.34945	0.082346
40	0.08667	0.36545	0.083039
50	0.08767	0.38145	0.083734
60	0.08867	0.39745	0.084429

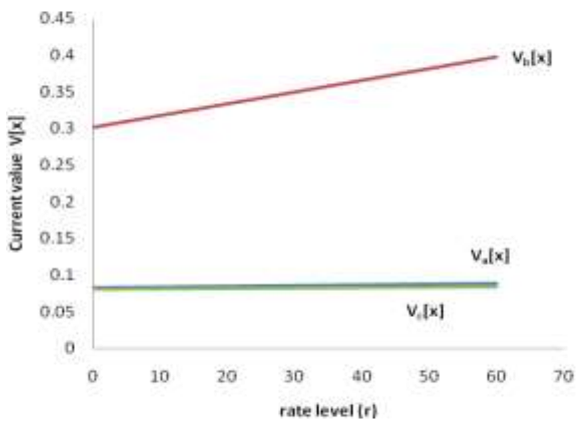

Figure 1 Show the expected optimal inventory level against time

Table 2 below shows the expected current values against the discount rate applied to consumption utility and Fig 2 shows the expected optimal inventory level against discounted rate

Table 2

γ	$V_a[x]$	$V_b[x]$	$V_c[x]$
0	0.460517	0.405045	0.080171
10	0.560517	0.430045	0.081171
20	0.660517	0.455045	0.082171
30	0.760517	0.480045	0.083171
40	0.860517	0.505045	0.084171
50	0.960517	0.530045	0.085171

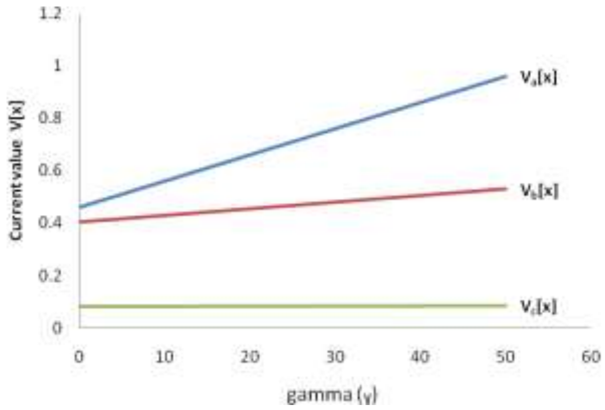

Figure 2

Table 3 below shows the total expected current values against the utility of consumption at consumption rate and Fig. 3 shows the expected optimal inventory level against the utility of consumption at consumption rate

Table 3

ρ	$V_a[x]$	$V_b[x]$	$V_c[x]$
1	21.30258	66.9957	58.6052
2	5.997866	17.5944	15.8992
3	3.02262	8.2537	7.679
4	1.92221	4.908	4.6854
5	1.3824	3.321	3.2429

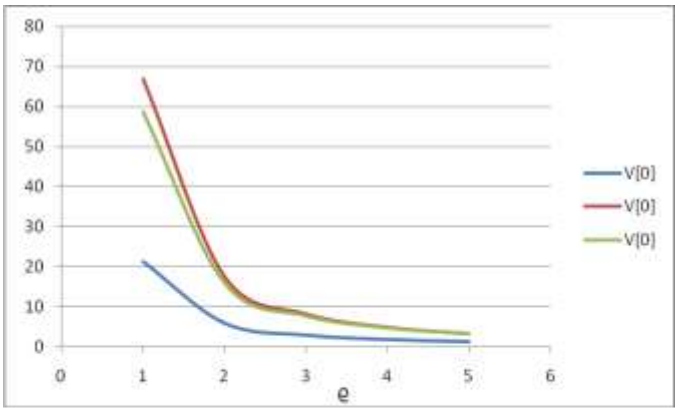

Figure 3

5.2. Discussion and Interpretation

From table 1 and Fig 1, as the current values showed against the rate level of the utility with consumption rate increases, then the expected optimal inventory level against time slightly increases. Likewise in table 2 and Fig. 2 as the expected current values against the discount rate applied to consumption utility then, the expected optimal inventory level against discounted rate also increases .Meanwhile in table 3 and Fig 3 indicates that the total expected current values against the utility of consumption at consumption rate decreases as then, the expected optimal inventory level against the utility of consumption at consumption rate also decreases

6. CONCLUSION

We have successfully examined the optimal stochastic control principle and its' application in formulating consumption model of Landmark University development ventures (LMDV) using Stochastic Differential Equations (SDE) as an ordinary differential equations (ODE) driven by white noise and we justified the connection between the Ito's integral and white noise in the case of non-random integrands interpreted as cost functions. The inventory level and consumption are seeing clearly to be stochastic in nature and demand rate is equally seeing to be deterministic in nature. Numerical illustrative examples used for stochastic consumption model were to displays the optimal expected inventory level E(x) against time (t) and the expected current values E(v) against time (t).

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