



A TWO-DIMENSIONAL SIMULATION OF ULTRASONIC WAVE PROPAGATION IN CONCRETE USING FINITE ELEMENT METHOD

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ABSTRACT

Due to the propagation and interaction characteristics in materials of the ultrasonic wave, it is widely used in the different fields such as medical, mechanical, and construction engineering, specially to detect cracks and defects in concrete construction, and to identify characteristics of concrete material as well. Understanding the propagation and interaction characteristics of the ultrasound wave in isotropic and anisotropy elastic materials plays a very important role. The paper presents a method for a two-dimensional simulation of ultrasonic wave propagation in concrete structure treating as homogenous, isotropic and anisotropic elastic media. A mathematical model of wave propagation using dynamic equilibrium equations in concrete structures with inclusions, steel reinforcements and cracks, is introduced. Then, numerical simulation is mentioned based on finite element method. The results of simulation show two-dimensional images of wave propagation inside concrete structures, as well as displacements, stresses and strains at specific points in the material according to space and time. The simulations are made for different cases of concrete structures with inclusions, steel reinforcements and open cracks.

Keywords: Simulation, ultrasonic wave propagation, finite element method, inclusion, steel reinforcement, crack.

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1. INTRODUCTION

Nowadays, ultrasonic testing (UT) is widely used in the different fields because it has many advantages such as fast, accurate and it can identify many characteristics of defects inside the test object. UT uses ultrasonic wave at high frequencies ranging from a few kilohertz to several hundred kilohertz [1].

Concrete materials consist of aggregates and cement, and other materials depending on the usage environment. Due to its complexity and furthermore when defects occur, ultrasonic wave propagation in concrete involves multiple scattering and mode conversions. Therefore, to use UT for concrete, it is necessary to understand the propagation mechanism of ultrasonic waves within this material, this leads to the importance of the simulation of ultrasonic wave propagation in concrete, in combining with experimentation.

There are two different approaches to simulate the propagation of ultrasonic waves in materials: finite difference method (FDM) [2, 3] and finite element method (FEM) [4, 5]. Based on these two methods, some researchers have improved some other techniques and methods such as Elastodynamic Finite Integration Technique (EFIT) [6, 7], finite integration technique (FIT) [1, 8], spectral element method (SEM) [9, 10], boundary finite element method [11, 12].

For concrete structures, F. Schubert and B. Koehler [7] use EFIT to simulate a diffusive reverberation measurement at a concrete specimen with aggregates, pores, taking into account of viscoelastic damping. K. Nakahata and al. [13] have proposed a tool for ultrasonic wave propagation in a target material with complex outer surface or various inclusions by combining FIT and FEM with an image-based modeling. In [1, 13], the authors investigated the concrete block at mesoscopic level. A. Van Pamel and al. [5, 9] have used FEM for solving three-dimensional elastodynamic propagation problem within media that present random variations in their elastic properties. However, these studies have not investigated the inside defects. Recently, Rucka et al. [9] have successfully simulated the propagation of ultrasonic waves in concrete with simple cracks. Mathematical frameworks are well established and developed, wave propagation in real engineering structures still remains an open area of research.

The paper deals with the modelling of the propagation of ultrasonic waves in complex concrete structures, homogeneous, isotropic and anisotropic elastic media by FEM. Simulations perform for many cases of concrete structures with inclusions, steel reinforcements and open cracks. A displacement excitation is applied perpendicularly to the free surface of the structure, to model the ultrasonic transmitting transducer. A program of simulating ultrasonic wave propagation in concrete is developed using the Matlab tools. The simulation results display images of two-dimensional wave propagation inside concrete, such as displacements, stresses and strains at specific points in concrete structures according to space and time. In addition, the paper also analyzes the effect of source frequency on the displacement value at the points in the concrete structure.

2. METHODOLOGY

2.1. Mathematical model of wave propagation

By applying the Hamilton principle, the equations of motion of the structure will be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0 \tag{1}$$

With $i = 1, 2, \dots, n$.

Where $L = T - \Pi$, L is Lagrangian function for the system, T is kinetic energy, Π is potential energy, $q = (q_1, q_2, \dots, q_N)$ are generalized coordinates.

Investigating a solid with a distributed volume as shown in Figure 1, the kinetic energy T is as follows:

$$T = \frac{1}{2} \int_V \dot{u}^T \dot{u} \rho dV \tag{2}$$

Where ρ is density, \dot{u} is velocity vector at point x $\dot{u} = \{\dot{u}, \dot{v}, \dot{w}\}^T$

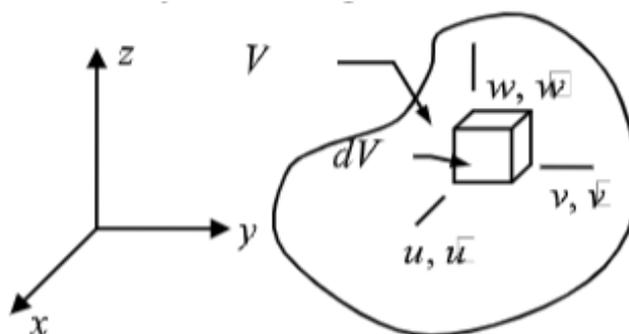


Figure 1 Solid with distributed volume

Step 1 In the FEM method, the object is divided into elements. In each element, the displacement u will be determined according to node displacements q by shape functions N

$$u = Nq \tag{3}$$

Step 2 In dynamic calculation, the displacement components q depends on the time, and the velocity vector is determined by

$$\dot{u} = N\dot{q} \tag{4}$$

Therefore, the kinetic energy of each element is

$$T = \frac{1}{2} \dot{q}^T \left[\int_e \rho N^T N dV \right] \dot{q} \tag{5}$$

The expression in brackets is called the element matrix of the element

$$m_e = \int_e \rho N^T N dV \tag{6}$$

The kinetic energy of the whole object is determined as follows:

$$T = \sum_e T_e = \sum_e \frac{1}{2} \dot{q}^T m_e \dot{q} = \frac{1}{2} \dot{Q}^T M \dot{Q} \tag{7}$$

Where Q is general nodal displacement matrix.

The total potential Π of an elastic object is the sum of the deformation energy U and the work W of external forces

$$\Pi = U + W \tag{8}$$

With a linear elastic object, the total deformation energy is determined by

$$U = \frac{1}{2} \int_V \sigma^T \varepsilon dV \quad (9)$$

Where σ , ε are respectively the stress and the strain of the element.

The work of external forces is defined by

$$W = - \int_V u^T P dV - \int_S u^T T dS - \sum_{i=1}^n u_i^T F_i \quad (10)$$

Where P is body force, T is surface force, u_i and F_i are displacement and concentrated force at the node i, respectively.

Hence, we can deduce

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F \quad (11)$$

By applying the Hamilton principle, we obtain the equation of motion

$$M \ddot{Q} + K Q = F \quad (12)$$

Where M is the mass matrix, K is the structural stiffness matrix, F is the vector of applied loads, Q is the general nodal displacement matrix. Damping is not considered in this study.

Next, the paper applies the finite element method to calculate stiffness matrices and mass matrices in the equation of motion.

2.2. Finite element formulation

For the two-dimensional simulation, the object is divided into triangle elements. The mass and stiffness matrix are determined as follows [14, 15]:

Stiffness matrix is

$$k_e = t_e A_e B^T D B \quad (13)$$

Where t_e is the element thickness, A_e is the area of the element, B is the deformation-displacement matrix of the element and defined as follows:

$$B = \frac{1}{\det J} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \quad (14)$$

D is the stress-strain matrix. This stress-strain relationship is divided into two cases: plane stress and plane strain.

In case of plane stress, D is defined as follows:

$$D = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (15a)$$

And in case of plane strain:

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (15b)$$

Where E and ν are elastic module and Poisson’s ratio of the material, respectively.

Mass matrix is written as follows:

$$m_e = \frac{\rho A_e t_e}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (16)$$

The general stiffness matrix K and general mass matrix M of the whole structure are concatenated from the stiffness and mass matrix of the elements.

Because the concrete structure has a significant thickness, the paper selected the plane strain case.

2.3. Newmark Method for differential equation solution

There are many methods to solve the equation of motion (12), defined in the previous section. This equation is solved by the Newmark method, because this method is a simple solution method and suitable for dynamic calculation.

Equation (12) is discretized in each time step at time (i+1) as follows:

$$M\ddot{Q}_{i+1} + KQ_{i+1} = F_{i+1} \quad (17)$$

The idea of this method is that the value at time i+1 is calculated from the value of the known solution at time i. The value of velocity and displacement at time i+1 is determined by equation [16]:

$$\begin{cases} Q_{i+1} = Q_i + \dot{Q}_i\Delta t + \left(\frac{1}{2} - \beta\right)\Delta t^2\ddot{Q}_i + \beta\Delta t^2\ddot{Q}_{i+1} \\ \dot{Q}_{i+1} = \dot{Q}_i + \ddot{Q}_i\Delta t + (1 - \gamma)\Delta t\ddot{Q}_i + \gamma\Delta t\ddot{Q}_{i+1} \end{cases} \quad (18)$$

Δt is time step; $\gamma = \frac{1}{2}, \beta = \frac{1}{4}$ are the parameters of the average acceleration in Newmark method. Substituting Equation (18) into (17), we get the value \ddot{Q}_{i+1} of acceleration at the end of time step i+1. Substituting \ddot{Q}_{i+1} into (18), we get the displacement value Q_{i+1} and velocity \dot{Q}_{i+1} at time step i+1.

2.4. Problem stability criterion

Stability is guaranteed by choosing an appropriate time step. If the time step is too small, the solution will require a large calculation time. On the other hand, if the time step is too large, the calculation can fail because of divergence. The stability condition is that the analysis time step Δt should be smaller than the critical time step Δt_{cr} . The paper assumes that the deformation of the element is small, the selected critical time step is the time that the longitudinal wave passes through the smallest element. Then, the time step is defined as follows:

$$\Delta t \leq \Delta t_{cr} = \frac{\Delta L}{c_L} \quad (19a)$$

Where ΔL is the smallest element size and c_L is the velocity of the longitudinal wave.

Figure 2 shows the value of ΔL for a linear square element, a linear equilateral-triangular element and a quadratic equilateral-triangular element.

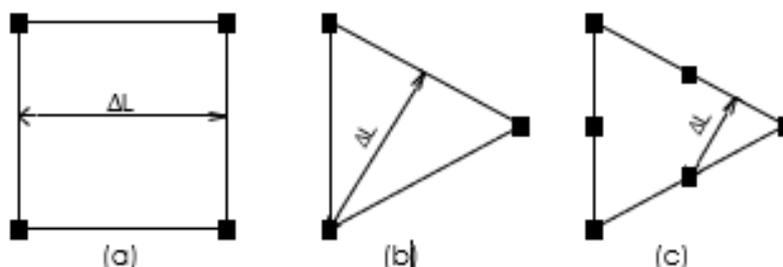


Figure 2 Illustration of ΔL for a) linear square element, b) linear triangle element and c) quadratic triangle element.

While the analysis time step should be strictly satisfied due to the stability, discrete space conditions seem less important. It seems reasonable that the smallest wavelength in the signal should be discrete with at least 8 grid points. This condition leads to the following formulas [17, 18]:

$$\Delta x \leq \frac{1}{8} \lambda_{\min} = \frac{1}{8} \frac{c_{\min}}{f_{\max}} \quad (19b)$$

where λ_{\min} is the shortest wavelength, c_{\min} is the lowest wave speed of the medium and f_{\max} denotes the highest frequency in the signal.

2.5. Initial and boundary conditions

2.5.1 Initial conditions

The medium is supposed to be in equilibrium at time $t=0$, therefore displacement, velocity and acceleration are set to zero everywhere in the medium.

For the point source, the excitation is applied to the displacement of the emission node. The displacement, the velocity and the acceleration at the emission node are defined as follows:

$$q = q_0 \sin(\omega t) \quad (20)$$

$$\frac{dq}{dt} = \dot{q} = v = \omega \cdot q_0 \cos(\omega t) \quad (21)$$

$$\frac{dv}{dt} = \ddot{q} = \omega^2 \cdot q_0 \sin(\omega t) \quad (22)$$

Where q is the displacement of emission node.

2.5.2 Boundary conditions

When solving equation of motion by finite element method, there are many types of boundary conditions [14, 15, 19]. In this paper, the boundary conditions of the four surveying cases: concrete block, concrete block with an inclusion, concrete block with an inside steel bar, concrete block with an open crack (figure 4) are determined as free surface.

2.6. Simulation algorithms

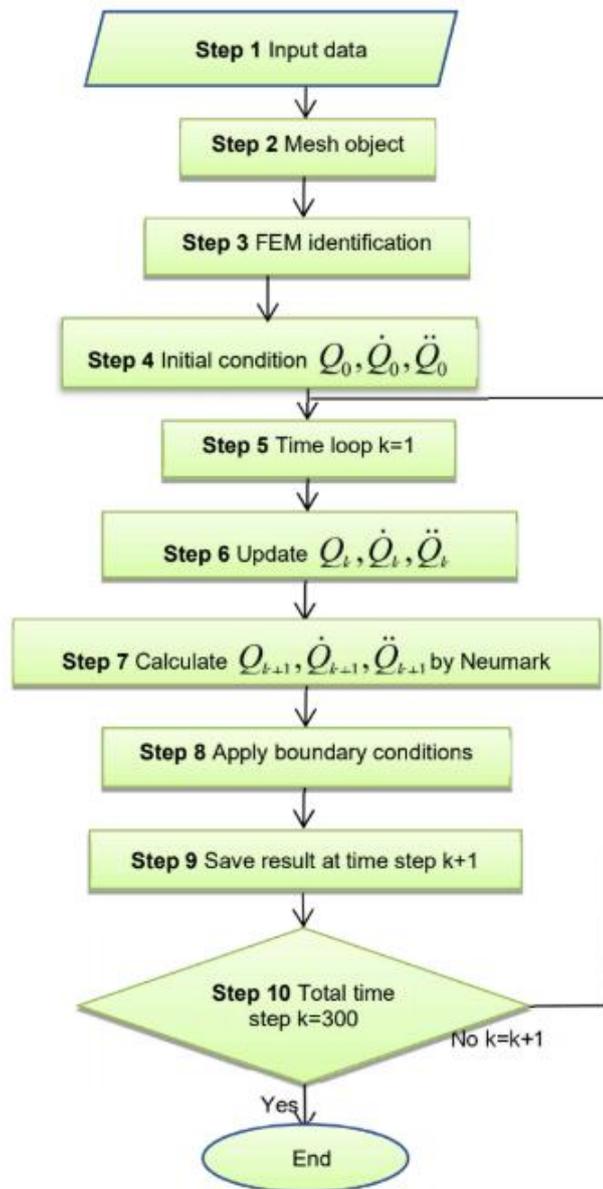


Figure 3 Simulation flowchart

The steps to simulate the ultrasonic wave propagation in concrete materials are performed as follows:

Step 1 Input data

The material is assumed as concrete with density $\rho=2500\text{kg/m}^3$, Young's modulus $E=10^9\text{N/m}^2$, Poisson's ratio $\nu=0.2$. The material properties of the steel bar are of density $\rho = 7800\text{kg/m}^3$, Young's modulus $E=2.10^9\text{N/m}^2$, Poisson's ratio $\nu = 0.2$. An ultrasonic wave with center frequency 50kHz and amplitude $q_0=0.1\text{mm}$ is radiated from a transducer contact region on the top surface of the concrete block as shown in Fig 4a. Newmark parameters $\gamma=1/2$ and $\beta=1/4$.

Step 2 Mesh object

According to the Eq. (19b), the chosen discrete mesh size is 3mm, then the time step is calculated according to the Eq. (19a) and the number of time steps is 300.

Step 3 FEM identification

Calculation of stiffness matrix and mass matrix in the equation of motion (12).

Step 4 Initial condition

Initialize displacement, velocity and acceleration of nodes at initial time.

Step 5 Time loop

The loop starts from the time step $k = 1$ to the last time step $k = 300$.

Step 6 Update displacement, velocity and acceleration at nodes at time step k .

Step 7 Calculate displacement, velocity and acceleration at nodes at time step $k+1$ by the Neumark method.

Step 8 Apply boundary conditions

Step 9 Save result at time step $k+1$

Step 10 Check the condition of the loop

If the time step is smaller than the total time step, then the program goes back to step 4, otherwise the loop ends. The simulation steps are represented by the flowchart in Figure 3.

3. NUMERICAL EXPERIMETATIONS

3.1. Matlab programing

Today, there are many computational tools that support numerical simulation. Among them, Matlab is a tool used in many fields, users can use the commands available or write new commands in accordance with the calculation requirements.

The basic steps of the workflow used for ultrasonic simulation in Matlab code can be separated into four parts:

- Pre-processing: Prepare necessary input parameters such as material properties, source excitation, mesh object, boundary conditions.
- Calculate stiffness matrix (K) and mass matrix (M) of the whole object in the equation of motion (12):
 - Calculate stiffness matrix (k_e) and mass matrix (m_e) of each element according to formulas (13) and (16)
 - For element step $k = 1$: n_e (n_e : number of elements in the structure), concatenate the general stiffness matrix (K) and general mass matrix (M) from k_e and m_e
- Simulation: Carry out the numerical ultrasonic wavefield simulation using FEM for given number of time steps (n_t)
 - Load all the input data and parameters
 - Initialize displacements, velocities and accelerations
 - For time step $k = 1$: n_t
 - Update displacements, velocities and accelerations at time step k
 - Update the boundary condition
 - Calculate displacements, velocities and accelerations at time step $k+1$ by the Neumark method
 - Save results at time step $k+1$

- Post-processing: Plot the displacement field, create animation snapshot and other tasks.

3.2. Structure geometry

The geometry of the sample concrete structure under investigation is presented in Fig 4. The concrete block has the following dimensions: length is 15cm, height is 15cm and thickness is 100cm.

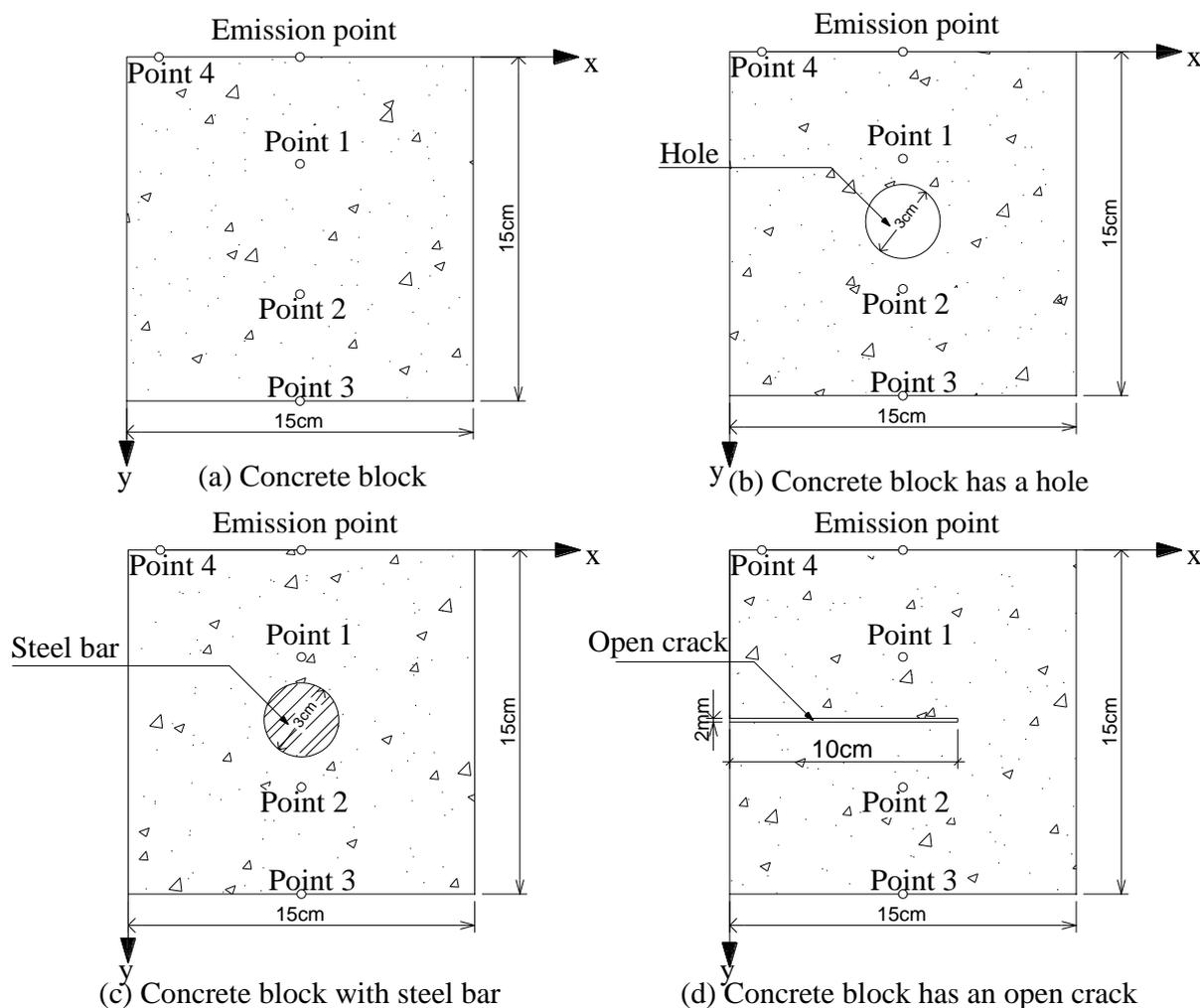


Figure 4 Four survey cases

We investigate the received signals at four points located within the material in different depths, their coordinates are as follows: point 1: $x=7.5\text{cm}$; $y=5\text{cm}$; point 2: $x=7.5\text{cm}$; $y=10\text{cm}$; point 3: $x=7.5\text{cm}$; $y=15\text{cm}$; point 4: $x=2.5\text{cm}$; $y=0\text{cm}$.

We survey the 4 cases as follows:

- Case 1: concrete block as shown in Fig. 4a
- Case 2: concrete block having a 3cm diameter cylindrical hole in the middle of the sample, as shown in Fig. 4b
- Case 3: concrete block with 3cm diameter steel bar in the middle of the sample as shown in Fig. 4c
- Case 4: concrete block with open crack, crack width is 2mm, as shown in Fig. 4d.

3.3. Simulation results and discussions

Fig. 5 shows two-dimensional FEM based images of wave propagation (nodal displacement field) inside the material of four survey cases at the same time. The images give an overview of wave propagation in different cases. Next, the paper will discuss the displacement value received at the receiving point in the survey cases.

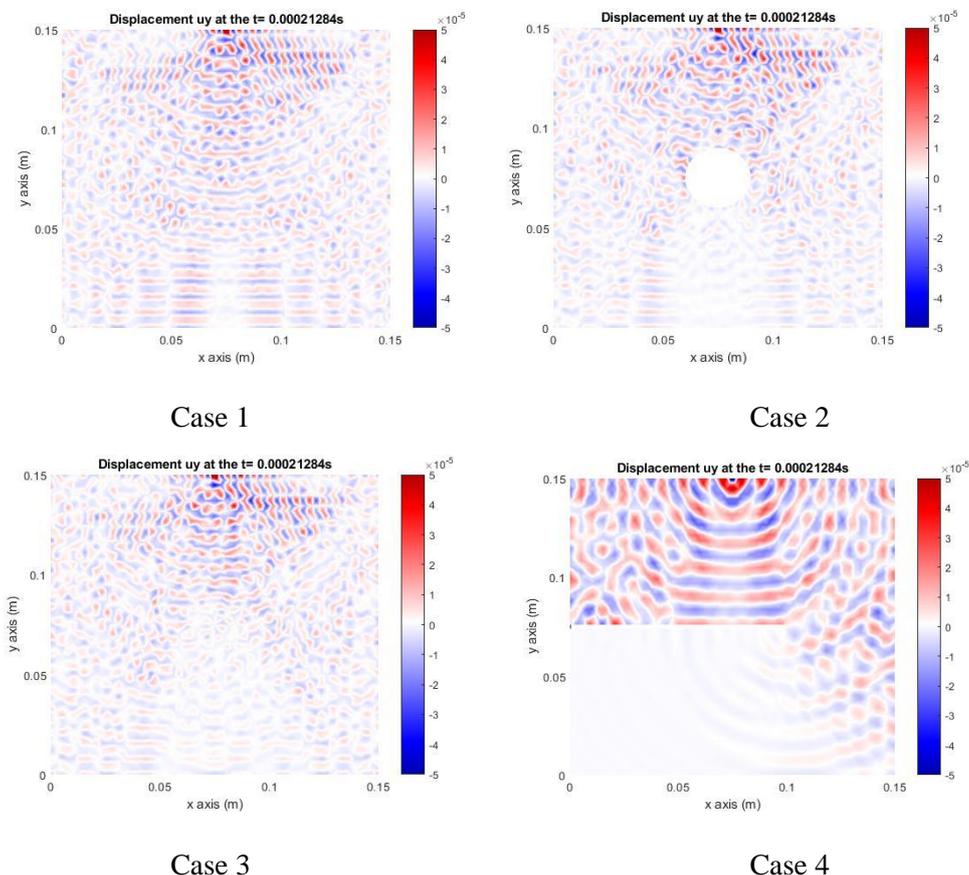


Figure 5 Two-dimensional images of wave propagation inside material of four survey cases, at the same time

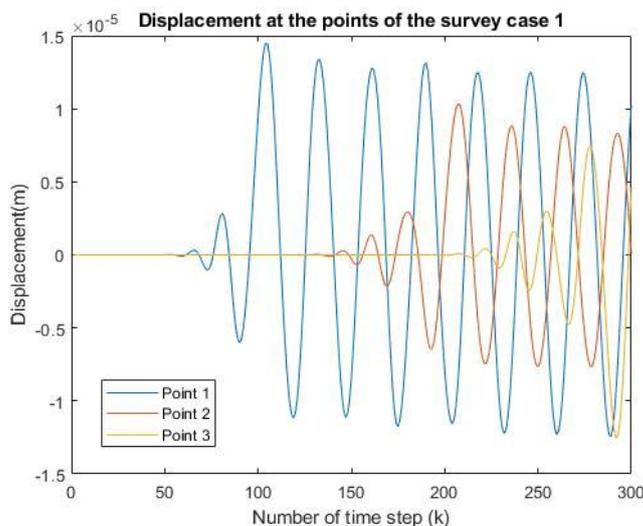


Figure 6 Displacement values for the survey case 1 at point 1, 2 and 3

For concrete samples (case 1), the paper determines displacement at three different points within the sample and the results are shown in Figure 6. This figure shows that the ultrasonic wave reaches point 1 first, then point 2 and point 3 successively. This is perfectly appropriate because points 1 and 2 are closer to the source than point 3.

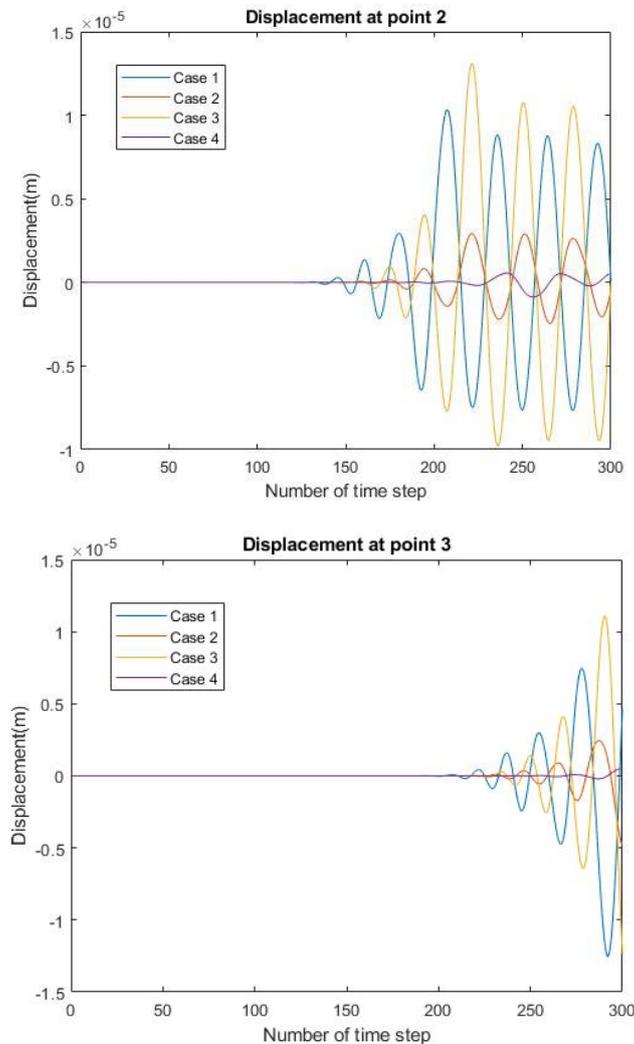


Figure 7 Displacement values at point 2 and point 3 for four survey cases.

Figure 7 gives the displacement results at the point 2 and point 3 for four survey cases. The results show that when ultrasonic wave propagates in different structure geometries, the displacement results are different. In other words, the propagation of ultrasonic wave depends on the wave propagation environment.

To investigate the differences in characteristics between transverse and longitudinal waves, the displacement values at point 1 and point 4 in Figure 8 are surveyed. The paper assumes that only the y-direction displacement is created at emission point, so the ultrasonic wave propagating to point 1 is the longitudinal wave and the wave propagating to point 4 is the transverse wave. Survey results are shown in Figure 9.

The results show different characteristics of transverse and longitudinal wave propagation. At point 1, the maximum amplitude is at the time step $k = 110$, while at point 4, the maximum amplitude is at time step $k = 140$. This means that longitudinal waves propagate faster than transverse waves. In addition, the amplitude of displacement value of the transverse wave is

about twice the value of the longitudinal wave. The above results are perfectly suitable because longitudinal waves propagate faster than transverse waves, whereas the transverse wave energy is larger than that of longitudinal waves. This explains why transverse waves are often used to check and evaluate material properties.

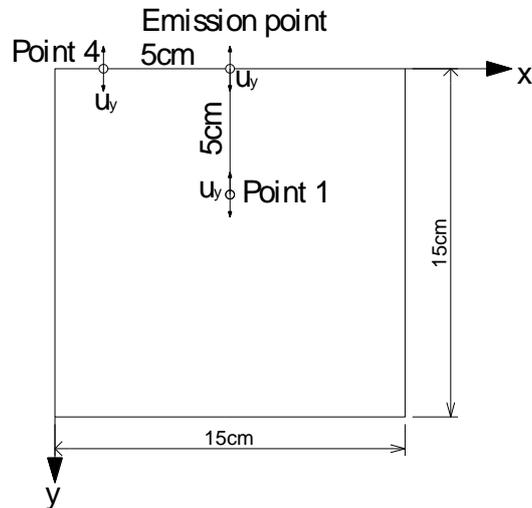


Figure 8 Location of emission point and receiving points

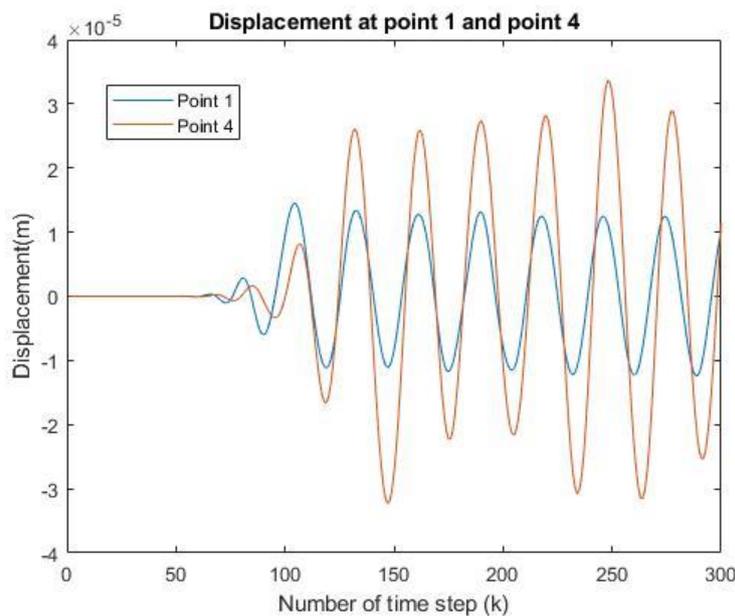


Figure 9 Displacement values at point 1 and point 4 in figure 8

3.4. Influence of ultrasonic frequency on attenuation of ultrasonic amplitude

In ultrasonic testing method, ultrasonic waves with high frequency are often used to identify the defects. To explain this, the paper investigates the influence of ultrasonic frequencies on the attenuation of the wave amplitude with two frequencies $f=50,000\text{Hz}$ and $f=100,000\text{Hz}$. The results of displacement values at point 2 with two different frequencies are shown in Figure 10 and table 1.

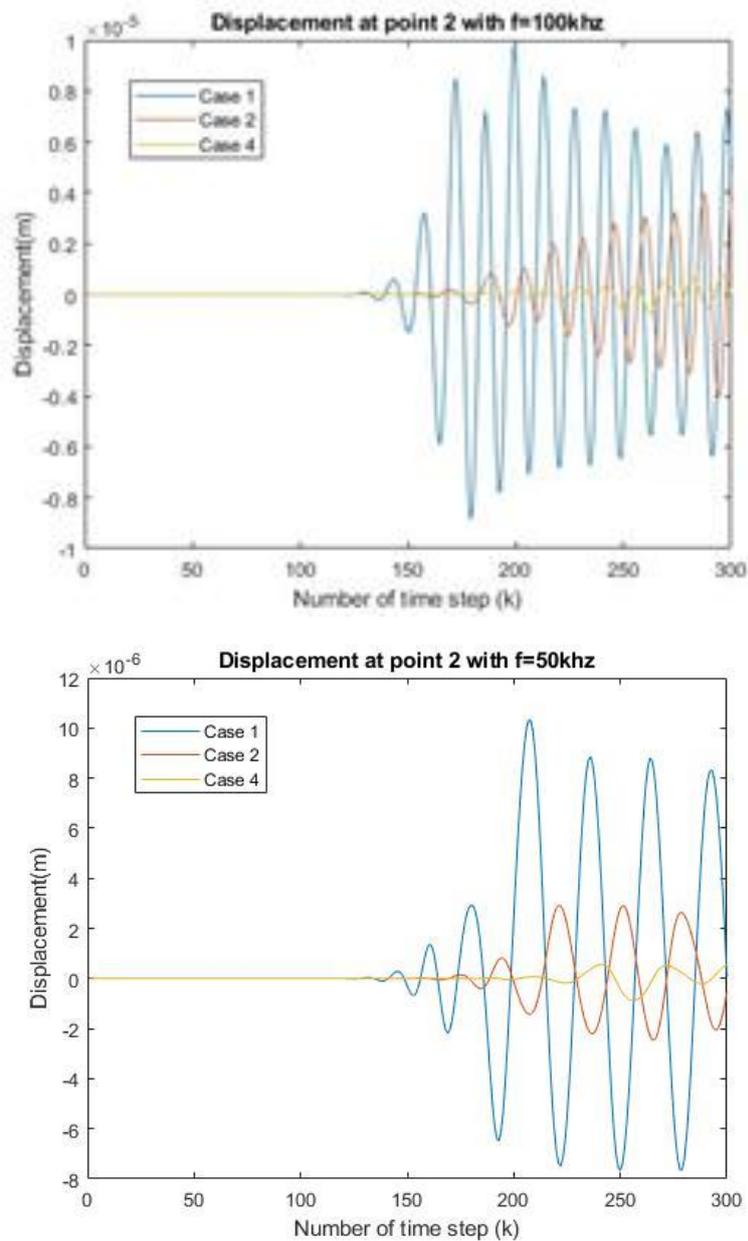


Figure 10 Influence of ultrasonic frequency on attenuation of ultrasonic amplitude

Table 1 Influence of ultrasonic frequency on attenuation of ultrasound amplitudes

Survey cases	f=50kHz	f=100kHz
Case 1	100%	100%
Case 2	30%	40%
Case 4	8%	10%

The results from figure 10 and table 1 show that the amplitude of wave displacement values of the sample cases with defects such as hole (case 2) and open crack (case 4) were significantly reduced compared to the sample without defects. For each case, higher the frequency is, lower

the degree of the attenuation of wave amplitude. This is an important characteristic, it explains why it is necessary to use high frequency ultrasonic wave.

4. CONCLUSION

The modelling of ultrasonic wave propagation in complex concrete structures, homogeneous, isotropic and anisotropic elastic medias by FEM was presented. Simulations were made for many cases of concrete structures with inclusion, steel reinforcement and open crack. A program for simulating ultrasonic wave propagation in concrete is developed using the Matlab tools. Simulation results show that the finite element method is well suited for simulating the propagation of ultrasonic waves within the material. The results of analyzing the characteristics of transverse and longitudinal waves and the effect of the emission frequency on the wave amplitude demonstrate the accuracy of the calculation program.

Based on these simulation results, further works in the future will apply the signal processing techniques to identify defects in concrete block.

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