NUMERICAL SOLUTION OF A CONTINUOUS MODEL OF ECONOMY USING MTSA-DERIVED BLOCK METHOD

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ABSTRACT:
Continuous models of economy adopted to either define evolutions of economic systems or investigate economic dynamics, amongst its other areas of applications, are known to be related to differential equations. The considered model in this article takes the form of a second order non-linear ordinary differential equation (ODE) which is conventional solved by reducing to the system of first order. This approach is computationally tasking, unlike block methods which bypasses reduction by directly solving the model. A new approach is introduced named Modified Taylor Series Approach (MTSA). Hence, the resultant MTSA-derived block method is implemented to solve the second order non-linear model of economy under consideration.

Key Words: Continuous Model, Economy, Modified Taylor Series Approach, Block Method, Second Order

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1. INTRODUCTION
Application of differential equations has evolved over the years in its modelling of change or motion in different areas of science. The availability and simultaneous growth of research in computational concepts has further made the theory of differential equations to be an essential tool of analysis, particularly in the field of economics. It would be difficult to comprehend the
contemporary literature of economics if one does not understand basic concepts and results of modern theory of differential equations. Hence, it is of great importance to investigate into the growing research in this area.

Basically, a differential equation expresses the rate of change of the current state as a function of the current state. Continuous models of economy also follow this concept, evident in examples such as investigating economic dynamics or defining evolutions of economic systems (Brock, 2018). These scenarios are closely connected with the differential equations in first order or higher order, especially ordinary differential equation (ODE), which consider the relation between a dependent variable and one independent variable. A simple first order ODE showing the changes of the Gross Domestic Product (GDP) with respect to time is given in Equation (1).

\[ x'(t) = gx(t) \] (1)

Where \( t \) denotes time and \( x'(t) \) is the derivative of the function \( x \) with respect to \( t \) (Zhang, 2005). The expression considers state \( x \) of the GDP of the economy, with the rate of change of the GDP being proportional to the current GDP. The growth rate of the GDP is \( x'/x \), such that, if the growth rate \( g \) is given at any time \( t \), the GDP at \( t \) is given by solving the differential equation. However, in order to accommodate more variables and produce a more realistic model, many differential equations encountered in economics involve higher order derivatives. An example is the continuous model under consideration in this article.

This second order non-linear differential equation model defined as

\[ Q^* = \alpha Q' \left( P + \frac{dP}{dQ} Q \right) \] (2)

Considers a study of the nature of growth of production in the conditions of the competition, where the quantity is defined as a function of time. Presenting the equation of balance of investments into production and research of speed of production gives rise to Equation (2) for finding of quantity of production of \( Q(t) \) and determining the nature of its increase (Vorontsova & Gorskaya, 2015). Conventionally, since the theory for systems of first-order equations is simple and the intuitive idea of what a differential equation means is clear, it is usually convenient to replace a higher-order equation such as Equation (2) by system of first order equations. Sadly, it consumes a lot of time and further affects the accuracy of the solution.

Block methods for the numerical solution of differential equations came to light in a bid to bypass the disadvantages of wastage in computational time of previously existing conventional methods (Awoyemi, 2003; Butcher, 2008). However, there are two major approaches that have been adopted in literature to develop the block methods; numerical integration approach and interpolation approach. In a recent work by Omar and Kuboye (2015), these two approaches were displayed, and the advantages and shortcomings discussed. It was stated that although the derivation using integration approach is more complicated in comparison to interpolation approach, this approach is able to generalize the formulation of the integration coefficients while the interpolation approach fails in this regard (Adeyeye & Omar, 2018). Lambert (1973) mentioned another approach for developing linear multistep methods which is the derivation by Taylor series. This approach is less rigorous to adopt as seen in studies by Li (2008) and Chen and Li (2012). However, the methods developed by the authors were restricted to the discrete schemes subject to boundary conditions, these methods were not further converted to block form. The advantage of block methods is the ability of the method to simultaneously evaluate the solution at different grid points for the equation under consideration. Hence, this
article considers the solution of second order non-linear ODE model using the block method derived via MTSA.

2. MODIFIED TAYLOR SERIES APPROACH (MTSA) FOR DEVELOPING $k$ -STEP BLOCK METHOD FOR SECOND ORDER ODES

Algorithm 1 shows the steps involved in adopting MTSA to develop block methods of any step-length $k$ for solving second order ODEs.

Algorithm 1

START

Step 1: Obtain the coefficients of the initial multistep scheme

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_{1,j} y_{n+j} + \sum_{j=0}^{k} \beta_{1,j} f_{n+j}$$

where $k$ is the step-length, $v = 1, 2$.

Step 2: Obtain the coefficients of the additional schemes

$$y_{n+1} = \sum_{j=0}^{k-1} \alpha_{2,j} y_{n+1,j} + \sum_{j=0}^{k} \beta_{2,j} f_{n+1,j}$$

$$y_{n+2} = \sum_{j=0}^{k-1} \alpha_{3,j} y_{n+2,j} + \sum_{j=0}^{k} \beta_{3,j} f_{n+2,j}$$

$$\vdots$$

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_{(k-1),j} y_{n+k,j} + \sum_{j=0}^{k} \beta_{(k-1),j} f_{n+k,j}$$

Step 3: Derive the coefficients of the first derivative schemes

$$y'_{n} = \sum_{j=0}^{k-1} \alpha_{k,j} y'_{n+j} + \sum_{j=0}^{k} \beta_{k,j} f_{n+j}$$

$$y'_{n+1} = \sum_{j=0}^{k-1} \alpha_{(k+1),j} y'_{n+1,j} + \sum_{j=0}^{k} \beta_{(k+1),j} f_{n+1,j}$$

$$y'_{n+2} = \sum_{j=0}^{k-1} \alpha_{(k+2),j} y'_{n+2,j} + \sum_{j=0}^{k} \beta_{(k+2),j} f_{n+2,j}$$

$$\vdots$$

$$y'_{n+k} = \sum_{j=0}^{k-1} \alpha_{(2k),j} y'_{n+k,j} + \sum_{j=0}^{k} \beta_{(2k),j} f_{n+k,j}$$

Step 4: Combine schemes obtained in Steps 1, 2 and 3 above to form a system of equations with matrix form equivalent $Ax = B$ where $x = (y_{n+1}, y'_{n+1}, \ldots, y_{n+k}, y'_{n+k})^T$

Step 5: Adopt matrix inverse approach to system of equations in Step 4 to obtain the expected block method.

STOP

Note that in Step 1 of Algorithm 1, the expected $\alpha_{1,j}$ are $\alpha_{4,k}$ and $\alpha_{6,k}$. 

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2.1. MTSA-Derived Two-Step Block Method for Second Order ODEs

The scheme for the two-step block method is given by

\[ y_{n+2} = \sum_{j=0}^{1} \alpha_{ij} y_{n+j} + \sum_{j=0}^{1} \beta_{ij} f_{n+j} \]  \hspace{1cm} (3)

Expanding individual terms in Equation (3) using Taylor series expansion about \( x = x_n \) and substituting the expansions back in Equation (3) gives

\[
y(x_n) + (2h)y'(x_n) + \frac{(2h)^2}{2!} y''(x_n) + \frac{(2h)^3}{3!} y'''(x_n) + \frac{(2h)^4}{4!} y^{(4)}(x_n) = \alpha_{10} y(x_n) \\
+ \alpha_{11} [y(x_n) + (h) y'(x_n) + \frac{(h)^2}{2!} y''(x_n) + \frac{(h)^3}{3!} y'''(x_n) + \frac{(h)^4}{4!} y^{(4)}(x_n)] + \beta_{10} y''(x_n) \\
+ \beta_{11} [y''(x_n) + (h) y'''(x_n) + \frac{(h)^2}{2!} y^{(4)}(x_n)] + \beta_{12} [y'''(x_n) + (2h) y^{(4)}(x_n) + \frac{(2h)^2}{2!} y^{(4)}(x_n)]
\]  \hspace{1cm} (4)

Rewriting Equation (4) in matrix form and equating coefficients of \( y^{(n)}(x_n) \) \( (n = 0, 1, \ldots, 4) \) yields

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & h & 0 & 0 & 0 \\
0 & \frac{(h)^2}{2!} & 1 & 1 & 1 \\
0 & \frac{(h)^3}{3!} & 0 & h & 2h \\
0 & \frac{(h)^4}{4!} & 0 & \frac{(2h)^2}{2!} & \frac{(2h)^3}{3!} \\
0 & \frac{(2h)^4}{4!} & & & \beta_{12}
\end{pmatrix}
\begin{pmatrix}
\alpha_{10} \\
\alpha_{11} \\
\beta_{10} \\
\beta_{11} \\
\beta_{12}
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2h \\
\frac{(2h)^2}{2!} \\
\frac{(2h)^3}{3!} \\
\frac{(2h)^4}{4!}
\end{pmatrix}
\]  \hspace{1cm} (5)

Adopting matrix inverse method, the values of \( \alpha_{10}, \alpha_{11}, \beta_{10}, \beta_{11} \) and \( \beta_{12} \) in Equation (5) are obtained as below

\[
(\alpha_{10}, \alpha_{11}, \beta_{10}, \beta_{11}, \beta_{12})^T = (-1, 2, \frac{h^2}{12}, \frac{5h^2}{6}, \frac{h^2}{12})^T
\]  \hspace{1cm} (6)

Substituting the obtained values in Equation (6) back in Equation (3) gives the scheme

\[ y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{12} (f_n + 10f_{n+1} + f_{n+2}) \]  \hspace{1cm} (7)

Since the block method being derived is to solve the second order non-linear ODE in Equation (2), the next step is to obtain the first derivative schemes at all grid points; \( x_n, x_{n+1}, x_{n+2} \).

The required schemes are

\[
y'_n = \frac{1}{h} (-y_n + y_{n+1}) + \frac{h}{24} (-7f_n - 6f_{n+1} + f_{n+2})
\]  \hspace{1cm} (8)

\[
y'_{n+1} = \frac{1}{h} (-y_n + y_{n+1}) + \frac{h}{24} (3f_n + 10f_{n+1} - f_{n+2})
\]  \hspace{1cm} (9)

\[
y'_{n+2} = \frac{1}{h} (-y_n + y_{n+1}) + \frac{h}{24} (f_n + 26f_{n+1} + 9f_{n+2})
\]  \hspace{1cm} (10)
Combining equations Equation (7) – (10) gives a system of equations which can be written in a matrix form

\[
\begin{pmatrix}
-2 & 1 & 0 & 0 & y'_{n+1} \\
-\frac{1}{h} & 0 & 0 & 0 & y'_{n+2} \\
-\frac{1}{h} & 0 & 1 & 0 & y''_{n+1} \\
-\frac{1}{h} & 0 & 0 & 1 & y''_{n+2}
\end{pmatrix}
= \begin{pmatrix}
y_{n} + \frac{h^2}{12} \left( f_{n} + 10 f_{n+1} + f_{n+2} \right) \\
y_{n} - \frac{1}{h} y'_{n} + \frac{h}{24} \left( -7 f_{n} - 6 f_{n+1} + f_{n+2} \right) \\
y_{n} - \frac{1}{h} y'_{n} + \frac{h}{24} \left( 3 f_{n} + 10 f_{n+1} - f_{n+2} \right) \\
y_{n} - \frac{1}{h} y'_{n} + \frac{h}{24} \left( f_{n} + 26 f_{n+1} + 9 f_{n+2} \right)
\end{pmatrix}
\]

Using matrix inverse method, \( y_{n+1}, y_{n+2}, y'_{n+1} \) and \( y'_{n+2} \) are determined in block form as

\[
y_{n+1} = y_{n} + h y'_{n} + \frac{h^2}{24} \left( 7 f_{n} + 6 f_{n+1} - f_{n+2} \right),
\]
\[
y_{n+2} = y_{n} + 2 h y'_{n} + \frac{h^2}{3} \left( 2 f_{n} + 4 f_{n+1} \right),
\]
\[
y'_{n+1} = y'_{n} + \frac{h}{12} \left( 5 f_{n} + 8 f_{n+1} - f_{n+2} \right),
\]
\[
y'_{n+2} = y'_{n} + \frac{2 h}{3} \left( f_{n} + 4 f_{n+1} + f_{n+2} \right).
\]

(11)

2.2. Convergence of the MTSA-Derived Two-Step Block Method

As conventionally known, a linear multistep method is convergent iff it is consistent and zero stable (Fatunla, 1988). This is also extended to block methods, and thus the consistency and zero-stability of the two-step block method is investigated.

**Definition 1:** A linear multistep method is consistent if it has order \( p \geq 1 \).

2.2.1. Order and Consistency of the MTSA-Derived Two-Step Block Method

Considering the linear operator associated with higher order ordinary differential equations defined as

\[
L \left[ y(x); h \right] = \sum_{j=0}^{k} \alpha_{j} y_{n+j} - \sum_{j=0}^{k} \beta_{j} f_{n+j} + \sum_{j=0}^{k} \lambda_{j} f'_{n+j},
\]

(12)

The individual terms of the correctors of the block method in Equation (11) are expanded using Taylor series expansion about \( x = x_{n} \) to give

\[
y(x_{n}) + (h) y'(x_{n}) + \frac{(h)^2}{2} y''(x_{n}) + \frac{(h)^3}{6} y'''(x_{n}) + \frac{(h)^4}{24} y^{(4)}(x_{n}) + \frac{(h)^5}{120} y^{(5)}(x_{n}) \\
y_{n} - hy'_{n} - \frac{h^2}{24} \left( 7 y''_{n} + 6 (y'''(x_{n}) + (h) y''(x_{n}) + \frac{(h)^2}{2} y'''(x_{n}) + \frac{(h)^3}{6} y^{(4)}(x_{n}) + \frac{(h)^4}{24} y^{(5)}(x_{n})) \right) \\
- (y''(x_{n}) + (2h) y'''(x_{n}) + \frac{(2h)^2}{21} y^{(4)}(x_{n}) + \frac{(2h)^3}{315} y^{(5)}(x_{n})) = 0.
\]

(13)

And

\[
y(x_{n}) + (2h) y'(x_{n}) + \frac{(2h)^2}{2} y''(x_{n}) + \frac{(2h)^3}{6} y'''(x_{n}) + \frac{(2h)^4}{24} y^{(4)}(x_{n}) + \frac{(2h)^5}{120} y^{(5)}(x_{n}) - y'_{n} \\
- 2hy'_{n} - \frac{h^2}{3} \left( 2 y''_{n} + 4 (y'''(x_{n}) + (h) y''(x_{n}) + \frac{(h)^2}{2} y'''(x_{n}) + \frac{(h)^3}{6} y^{(4)}(x_{n}) + \frac{(h)^4}{24} y^{(5)}(x_{n})) \right) = 0.
\]

(14)

Equation (13) and (14) take following form

\[
L \left[ y(x); h \right] = C_{0} y(x_{n}) + C_{1} h y'(x_{n}) + \ldots + C_{\rho} h^{p} y^{(p)}(x_{n}) + \ldots
\]

(15)
Thus, comparing the coefficients of \( h^n y^{(n)}_n \) in Equation (13) and (14) and rewriting in merged form gives

\[
\begin{align*}
(1,1,\frac{(2)^3}{2^3},\frac{(3)^4}{3^3},\frac{(4)^5}{4^5})^T - (1,0,0,0,0)^T - \frac{1}{24} [7(0,0,1,0,0,0)^T \\
+6\left(0,0,1,\frac{(2)^3}{2^3},\frac{(3)^4}{3^3}\right)^T - \left(0,0,1,2,\frac{(2)^2}{3^3},\frac{(3)^3}{3^3}\right)^T] = (0,0,0,0,0,\frac{1}{24})^T
\end{align*}
\]

and

\[
\begin{align*}
(1,2,\frac{(2)^3}{2^3},\frac{(3)^4}{3^3},\frac{(4)^5}{4^5})^T - (1,0,0,0,0,0)^T - 2(0,1,0,0,0,0)^T - \frac{1}{4} [2(0,0,1,0,0,0)^T \\
+4\left(0,0,1,\frac{(2)^3}{2^3},\frac{(3)^4}{3^3}\right)^T] = (0,0,0,0,0,\frac{1}{36})^T
\end{align*}
\]

The method is said to be of order \( p \) if \( C_0 = C_1 = \ldots = C_p = C_{p+1} = 0 \), \( C_{p+2} \neq 0 \) and \( C_{p+2} \) is the error constant. Hence, the order of the two-step block method is \( p = 3 \) with error constant \( C_5 = \left(\frac{1}{45},\frac{2}{35}\right)^T \).

With reference to Definition 1, the two-step block method is consistent.

### 2.2.2. Zero-Stability of the MTSA-Derived Two-Step Block Method

To analyze the two-step block method for zero stability, the correctors of the block method in Equation (11) are normalized to give the first characteristic polynomial \( \rho(r) \) as

\[
\rho(r) = \text{det} \left( R_j A^0 - A^1 \right) = \text{det} \left[ \begin{array}{cc}
  r & 0 \\
  0 & r^2 \\
\end{array} \right] \left[ \begin{array}{cc}
  1 & 0 \\
  0 & 1 \\
\end{array} \right] - \left[ \begin{array}{cc}
  0 & 1 \\
  0 & 1 \\
\end{array} \right]
\]

The roots of \( \rho(r) = 0 \) satisfy \( |r_j| \leq 1 \). Therefore, the block method is zero-stable.

Thus convergent, since it is both consistent and zero-stable.

### 3. Numerical Solution of the Model

The model as defined by the non-linear second order ODE in Equation (2) concerns the study of the nature of production growth in the conditions of competition. This section presents its numerical solution using the MTSA-derived block method. Following the approach stated in Vorontsova and Gorskaya (2015), the dependence of \( P(Q) \) is accepted in Equation (2) in the form of the linear function \( P(Q) = a - bQ \). This is assumed for simplicity, which implies that \( \frac{dP}{dQ} = -b \). Hence, transforming the model in Equation (2) to the non-linear form:

\[
Q^* = \alpha Q(a - 2bQ)
\]

The parameters are chosen as \( \alpha = 2, a = 4 \) and \( b = 1 \) and the results are displayed in Table 1.
Table 1. Solution Comparison of Equation (2)

<table>
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<tr>
<th></th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
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</tr>
</tbody>
</table>

Table 1 is in presented five columns. The first column gives the \( t \) – values, while the second and third columns presents \( Q(t) \) and \( Q'(t) \) respectively from Vorontsova and Gorskaya (2015) using Runge-Kutta’s method of fourth order. Columns four and five are for the results obtained via the block method, where \( Q(t) \) values are in column four while \( Q'(t) \) is in column five. It is observed that the solutions are in agreement, although there is no analytical solution to compute errors. To display the pattern of solution, a larger interval is considered for the block method solution as displayed in Figure 1.

Figure 1: Plot of the solution of \( Q \) and \( Q' \) for Equation (2) Model using Block Method
4. CONCLUSION

Application of differential equations in various fields such as economics is seen to have grown to be of importance over time. For some of these equations of continuous models of economy, approximate methods are required as analytic solutions are unavailable. Thus, the reason why suitable approach to approximate solutions of the arising differential equation models is important. The choice of approach to solve these differential equations approximately or numerically requires consideration of certain factors. Block methods have proved to be a more suitable concept as it is more convenient than reducing to a system of first order, before applying suitable numerical methods such as Runge-Kutta. The usability of a two-step block method is shown in this article to solve a continuous model of economy.

REFERENCES


