TWO FLUID ELECTROMAGNETO CONVECTIVE FLOW AND HEAT TRANSFER BETWEEN VERTICAL WAVY WALL AND A PARALLEL FLAT WALL

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ABSTRACT

The mixture of viscous and magneto convective flow and heat transfer between a long vertical wavy wall and a parallel flat wall in the presence of applied electric field parallel to gravity, magnetic field normal to gravity in the presence of source or sink is investigated. The non-linear equations governing the flow are solved using the linearization technique. The effect of Grashof number and width ratio is to promote the flow for both open and short circuits. The effect of Hartmann number is to suppress the flow, the effect of source is to promote and the effect of sink is to suppress the velocity for open and short circuits. Conducting ratio decreases the temperature whereas width ratio increases the temperature.

Keywords: Convection, Vertical Wavy, Electrically Conducting Fluid

1. INTRODUCTION

From a technological point of view the free convection study of fluids with known physical properties, which are confined between two parallel walls, is always important, for it can reveal hitherto-unknown properties of fluids of practical interest. In such buoyancy driven flows the exact governing equations are unwieldy so that recourse to approximations such as Boussinesq’s is generally called for. Assuming the linear density temperature variation (Boussinesq’s approximation) Ostrach [1] studied the laminar natural convection heat transfer in fluids with and without heat sources, in channels with constant wall temperature.

Previous studies of natural convection heat and mass transfer have focused mainly on a flat plate or regular ducts. Botfemanne [2] has considered simultaneous heat and mass transfer by free convection along a vertical flat plate only for steady state theoretical solutions with Pr = 0.71 and Sc
= 0.63. Callahan and Marner [3] studied the free convection with mass transfer on a vertical flat plate with \( \text{Pr} = 1 \) and a realistic range of Schmidt number.

It is necessary to study the heat and mass transfer from an irregular surface because irregular surfaces are often present in many applications. It is often encountered in heat transfer devices to enhance heat transfer. For examples, flat-plate solar collectors and flat-plate condensers in refrigerators. The natural convection heat transfer from an isothermal vertical wavy surface was first studied by Yao [4, 5, 6] and using an extended Prandtl’s transposition theorem and a finite-difference scheme. He proposed a simple transformation to study the natural convection heat transfer from isothermal vertical wavy surfaces, such as Sinusoidal surface.

The fluid mechanics and the heat transfer characteristics of the generator channel or significantly influenced by the presence of the magnetic field. Shail [7] studied the problem of Hartmann flow of a conducting fluid in a horizontal channel of insulated plates with a layer of non-conducting fluid over lying a conducting fluid in two fluid flow and predicted that the flow rate can be increases to the order of 30% for suitable values of ratio of viscosities, width of the two fluids with appropriate Hartmann number.

Hartmann [8] carried out the pioneer work on the study of steady magnetohydrodynamic channel flow of a conducting fluid under a uniform magnetic field transverse to an electrically insulated channel wall. Later Osterle and Young [9] investigated the effect of viscous and Joule dissipation on hydromagnetic force convection flows and heat transfer between two vertical plates with transverse magnetic field under short circuit condition. Poots [10] treated the open circuit case with and without heat sources, the two bounding plates being maintained at different temperatures, Umavathi [11] studied the effect of viscous and ohmic dissipation on magnetoconvection in a vertical enclosure for open and short circuits, recently Malashetty et.al. [11] studied the magnetoconvection of two-immiscible fluids in vertical enclosure.

Electric current in semi-conducting fluids such as glass and electrolyte generate Joulean heating Song [12], other application include those involving exothermic and endothermic chemical reaction and dealing with dissociating fluids Vajravelu and Nayefeh [13].

All the reference mentioned above dealt with vertical parallel walls. Since flow past wavy boundaries are encountered in many situation such as the rippling melting surface, physiological applications and many more as mention above, motivated us to study flow and heat transfer of the mixture of viscous and electrically conducting fluid in the presence of applied electric and magnetic fields between vertical wavy walls and a parallel flat wall.

2. MATHEMATICAL FORMULATION

![Fig. 1. Physical Configuration](image-url)
Consider the channel as shown in Fig. 1, in which the X axis is taken vertically upwards and parallel to the flat wall while the Y axis is taken perpendicular to it in such a way that the wavy wall is represented by $Y = -h^{(1)} + \epsilon \times \cos KX$ and the flat wall by $Y = h^{(2)}$. The region $0 \leq y \leq h^{(1)}$ is occupied by viscous fluid of density $\rho_1$, viscosity $\mu_1$, thermal conductivity $k_1$ and the region $h^{(2)} \leq y \leq 0$ is occupied another viscous fluid of density $\rho_2$, viscosity $\mu_2$, thermal conductivity $k_2$. The wavy and flat walls are maintained at constant and different temperatures $T_w$ and $T_1$ respectively. We make the following assumptions:

(i) that all the fluid properties are constant except the density in the buoyancy-force term;
(ii) that the flow is laminar, steady and two-dimensional;
(iii) that the viscous dissipation and the work done by pressure are sufficiently small in comparison with both the heat flow by conduction and the wall temperature;
(iv) that the wavelength of the wavy wall, which is proportional to $1/K$, is large.

Under these assumptions, the equations of momentum, continuity and energy which govern steady two-dimensional flow and heat transfer of viscous incompressible fluids are

**Region –I**

\[
\rho^{(1)} \left( \frac{U^{(1)}}{\partial X} + \frac{V^{(1)}}{\partial Y} \right) = \frac{\partial P^{(1)}}{\partial X} + \mu^{(1)} \nabla^2 U^{(1)} - \rho^{(1)} g \tag{1}
\]

\[
\rho^{(1)} \left( \frac{U^{(1)}}{\partial X} + \frac{V^{(1)}}{\partial Y} \right) = \frac{\partial P^{(1)}}{\partial Y} + \mu^{(1)} \nabla^2 V^{(1)} \tag{2}
\]

\[
\frac{\partial U^{(1)}}{\partial X} + \frac{\partial V^{(1)}}{\partial Y} = 0 \tag{3}
\]

\[
\rho^{(1)} C_p^{(1)} \left( \frac{U^{(1)}}{\partial X} + \frac{V^{(1)}}{\partial Y} \right) = k^{(1)} \nabla^2 T^{(1)} + Q_1 \tag{4}
\]

**Region –II**

\[
\rho^{(2)} \left( \frac{U^{(2)}}{\partial X} + \frac{V^{(2)}}{\partial Y} \right) = \frac{\partial P^{(2)}}{\partial X} + \mu^{(2)} \nabla^2 U^{(2)} - \rho^{(2)} g - \sigma B_o^2 U^{(2)} - \sigma B_o E_0 \tag{5}
\]

\[
\rho^{(2)} \left( \frac{U^{(2)}}{\partial X} + \frac{V^{(2)}}{\partial Y} \right) = \frac{\partial P^{(2)}}{\partial Y} + \mu^{(2)} \nabla^2 V^{(2)} \tag{6}
\]

\[
\frac{\partial U^{(2)}}{\partial X} + \frac{\partial V^{(2)}}{\partial Y} = 0 \tag{7}
\]

\[
\rho^{(2)} C_p^{(2)} \left( \frac{U^{(2)}}{\partial X} + \frac{V^{(2)}}{\partial Y} \right) = k^{(2)} \nabla^2 T^{(2)} + Q_2 \tag{8}
\]

where the superscript indicates the quantities for regions I and II, respectively. To solve the above system of equations, one needs proper boundary and interface conditions. We assume $C_p^{(1)} = C_p^{(2)}$.
The physical hydrodynamic conditions are

\begin{align*}
U^{(1)} &= 0 & V^{(1)} &= 0, & \text{at } Y = -h^{(1)} + \varepsilon \text{cosKX} \\
U^{(2)} &= 0 & V^{(2)} &= 0, & \text{at } Y = h^{(2)} \\
U^{(1)} &= U^{(2)} & V^{(1)} &= V^{(2)}, & \text{at } Y = 0 \\
\mu^{(1)} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right)^{(1)} &= \mu^{(2)} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right)^{(2)}, & \text{at } Y = 0 (9)
\end{align*}

The boundary and interface conditions on temperature are

\begin{align*}
T^{(1)} &= T_w & \text{at } Y = -h^{(1)} + \varepsilon \text{cosKX} \\
T^{(2)} &= T_i & \text{at } Y = h^{(2)} \\
T^{(1)} &= T^{(2)} & \text{at } Y = 0 \\
\kappa^{(1)} \left( \frac{\partial T}{\partial Y} + \frac{\partial T}{\partial X} \right)^{(1)} &= \kappa^{(2)} \left( \frac{\partial T}{\partial Y} + \frac{\partial T}{\partial X} \right)^{(2)}, & \text{at } Y = 0 (10)
\end{align*}

The conditions on velocity represent the no-slip condition and continuity of velocity and shear stress across the interface. The conditions on temperature indicate that the plates are held at constant but different temperatures and continuity of heat and heat flux at the interface.

The basic equations (1) to (8) are made dimensionless using the following transformations

\begin{align*}
(x, y)^{(1)} &= \frac{j}{h^{(1)}} (X, Y)^{(1)} ; & (u, v)^{(1)} &= \frac{j^{(1)}}{\nu^{(1)}} (U, V)^{(1)} \\
(x, y)^{(2)} &= \frac{j}{h^{(2)}} (X, Y)^{(2)} ; & (u, v)^{(2)} &= \frac{j^{(2)}}{\nu^{(2)}} (U, V)^{(2)} \\
\theta^{(1)} &= \frac{T^{(1)} - T_i}{T_w - T_S} ; & \theta^{(2)} &= \frac{T^{(2)} - T_i}{T_w - T_S} \\
\bar{\rho}^{(1)} &= \frac{\rho^{(1)}}{h^{(1)}}^2 ; & \bar{\rho}^{(2)} &= \frac{\rho^{(2)}}{h^{(2)}}^2 (11)
\end{align*}

where \( T_S \) is the fluid temperature in static conditions.

**Region –I**

\begin{align*}
\frac{\partial u^{(1)}}{\partial x} + \frac{\partial u^{(1)}}{\partial y} = - \frac{\partial \bar{\rho}^{(1)}}{\partial x} + \frac{\partial^2 u^{(1)}}{\partial x^2} + \frac{\partial^2 u^{(1)}}{\partial y^2} + G \theta^{(1)} (12)
\end{align*}

\begin{align*}
\frac{\partial v^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = - \frac{\partial \bar{\rho}^{(1)}}{\partial y} + \frac{\partial^2 v^{(1)}}{\partial x^2} + \frac{\partial^2 v^{(1)}}{\partial y^2} (13)
\end{align*}
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]
(14)

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{Pr} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \frac{\alpha}{Pr}
\]
(15)

**Region –II**

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial \bar{P}}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta m^2 r^2 h^4 G \theta^2 - M^2 m^2 u - m^2 h^4 r M^2 E
\]
(16)

\[
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial \bar{P}}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}
\]
(17)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]
(18)

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{km}{Pr} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \frac{m h^2 \alpha Q}{Pr}
\]
(19)

The dimensionless form of equation (9) and (10) using (11) become

\[
u^{(1)} = 0; \quad \nu^{(2)} = 0
\]
at \( y = -1 + \epsilon \cos \lambda x \)

\[
u^{(1)} = \frac{1}{rm^2} u^{(2)}; \quad \nu^{(2)} = \frac{1}{rm^2} v^{(2)}
\]
at \( y = 0 \)

\[
\left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{rm^2 h^2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]
at \( y = 0 \)

\[
\theta^{(1)} = 1
\]
at \( y = -1 + \epsilon \cos \lambda x \)

\[
\theta^{(2)} = \bar{\theta}
\]
at \( y = 1 \)

\[
\theta^{(1)} = \theta^{(2)}
\]
at \( y = 0 \)

\[
\left( \frac{\partial \theta}{\partial y} + \frac{\partial \theta}{\partial x} \right) = \frac{k}{h} \left( \frac{\partial \theta}{\partial y} + \frac{\partial \theta}{\partial x} \right)
\]
at \( y = 0 \)

**3. SOLUTIONS**

The governing equations (12) to (15), (16) to (19) and (24) to (25) along with boundary and interface conditions (20) to (21) are two dimensional, nonlinear and coupled and hence closed form solutions can not be found. However approximate solutions can be found using the method of regular perturbation. We take flow field and the temperature field to be
Region-I

\[ u^{(1)}(x, y) = u_0^{(1)}(y) + \varepsilon u_1^{(1)}(x, y) \]
\[ v^{(1)}(x, y) = \varepsilon v_1^{(1)}(x, y) \]
\[ P^{(1)}(x, y) = P_0^{(1)}(x) + \varepsilon P_1^{(1)}(x, y) \]
\[ \theta^{(1)}(x, y) = \theta_0^{(1)}(y) + \varepsilon \theta_1^{(1)}(x, y) \]

Region -II

\[ u^{(2)}(x, y) = u_0^{(2)}(y) + \varepsilon u_1^{(2)}(x, y) \]
\[ v^{(2)}(x, y) = \varepsilon v_1^{(2)}(x, y) \]
\[ P^{(2)}(x, y) = P_0^{(2)}(x) + \varepsilon P_1^{(2)}(x, y) \]
\[ \theta^{(2)}(x, y) = \theta_0^{(2)}(y) + \varepsilon \theta_1^{(2)}(x, y) \]

where the perturbations \( u_1^{(i)}, v_1^{(i)}, P_1^{(i)} \) and \( \theta_1^{(i)} \) for \( i=1,2 \) are small compared with the mean or the zeroth order quantities. Using equation (5.3.1) and (5.3.2) in the equation (12) to (15), (16) to (19) and (24) to (25) become

Zeroth order:

Region-I

\[ \frac{d^2 u_0^{(1)}}{dy^2} = -G \theta_0^{(1)} \]  
(24)
\[ \frac{d^2 \theta_0^{(1)}}{dy^2} = -\alpha \]  
(25)

Region-II

\[ \frac{d^2 u_0^{(2)}}{dy^2} - mh^2 M^2 u_0^{(2)} = m^2 rh^3 M^2 E - \beta m^2 r^2 h^2 G \theta_0^{(2)} \]  
(26)
\[ \frac{d^2 \theta_0^{(2)}}{dy^2} = -\frac{\alpha Q h^2}{k} \]  
(27)

First order:

Region-I

\[ \frac{u_0^{(1)}}{\partial x} + \frac{v_1^{(1)}}{\partial y} \frac{du_0^{(1)}}{dy} = - \frac{\partial P_1^{(1)}}{\partial x} + \frac{\partial^2 u_1^{(1)}}{\partial x^2} + \frac{\partial^2 u_1^{(1)}}{\partial y^2} + G \theta_1^{(1)} \]  
(28)
\[ \frac{u_0^{(1)}}{\partial x} = - \frac{\partial P_1^{(1)}}{\partial y} + \frac{\partial^2 v_1^{(1)}}{\partial x^2} + \frac{\partial^2 v_1^{(1)}}{\partial y^2} \]  
(29)
\[ \frac{\partial u_1^{(1)}}{\partial x} + \frac{\partial^2 v_1^{(1)}}{\partial y^2} = 0 \] (30) \\
\[ \Pr \left( u_0^{(1)} \frac{\partial \theta_1^{(1)}}{\partial x} + v_1^{(1)} \frac{\partial \theta_1^{(1)}}{\partial y} \right) = \frac{\partial^2 \theta_1^{(1)}}{\partial x^2} + \frac{\partial^2 \theta_1^{(1)}}{\partial y^2} \] (31) \\

**Region-II**

\[ u_0^{(2)} \frac{\partial u_1^{(2)}}{\partial x} + v_1^{(2)} \frac{\partial u_1^{(2)}}{\partial y} = -\frac{\partial P_1^{(2)}}{\partial x} + \frac{\partial^2 u_1^{(2)}}{\partial x^2} + \frac{\partial^2 u_1^{(2)}}{\partial y^2} + \beta m^2 r^2 h^3 G \theta_1^{(2)} - mh^2 M^2 u_1^{(2)} \] (32) \\
\[ u_0^{(2)} \frac{\partial \theta_1^{(2)}}{\partial x} = -\frac{\partial P_1^{(2)}}{\partial y} + \frac{\partial^2 v_1^{(2)}}{\partial x^2} + \frac{\partial^2 v_1^{(2)}}{\partial y^2} \] (33) \\
\[ \frac{\partial u_1^{(2)}}{\partial x} + \frac{\partial^2 v_1^{(2)}}{\partial y^2} = 0 \] (34) \\
\[ u_0^{(2)} \frac{\partial \theta_1^{(2)}}{\partial x} + v_1^{(2)} \frac{\partial \theta_1^{(2)}}{\partial y} = \frac{km}{\Pr} \left( \frac{\partial^2 \theta_1^{(2)}}{\partial x^2} + \frac{\partial^2 \theta_1^{(2)}}{\partial y^2} \right) \] (35) \\

where \( C^{(1)} = \frac{\partial \left( P_0 - P_S \right)^{(1)}}{\partial x} \) and \( C^{(2)} = \frac{\partial \left( P_0 - P_S \right)^{(2)}}{\partial x} \) taken equal to zero (see Ostrach, 1952). With the help of (5.3.1) and (5.3.2) the boundary and interface conditions (20) to (21) become

\[ u_0^{(1)} (-1) = 0 \] at \( y = -1 \) \\
\[ u_0^{(2)} (1) = 0 \] at \( y = 1 \) \\
\[ u_0^{(1)} = \frac{1}{rnh} u_0^{(2)} \] at \( y = 0 \) \\
\[ \frac{du_0^{(1)}}{dy} = \frac{1}{rnh} \frac{du_0^{(2)}}{dy} \] at \( y = 0 \) (36) \\
\[ \theta_0^{(1)} (-1) = 1 \] at \( y = -1 \) \\
\[ \theta_0^{(2)} (1) = \bar{\theta} \] at \( y = 1 \) \\
\[ \theta_0^{(1)} = \theta_0^{(2)} \] at \( y = 0 \) \\
\[ \frac{d\theta_0^{(1)}}{dy} = \frac{k}{h} \frac{d\theta_0^{(2)}}{dy} \] at \( y = 0 \) (37) \\
\[ u_1^{(1)} (-1) = -Cos \lambda x \frac{du_0^{(1)}}{dy} ; \quad v_1^{(1)} (-1) = 0 ; \quad u_1^{(2)} (1) = 0; \quad v_1^{(2)} (1) = 0 ; \] at \( y = -1 \) \\
\[ u_1^{(2)} (1) = 0; \quad v_1^{(2)} (1) = 0 ; \] at \( y = 1 \)
Introducing the stream function \( \psi \) defined by

\[
\psi = \psi (x, y) = \psi_1 (x, y) + \psi_2 (x, y)
\]

and eliminating \( P_1 \) and \( P_2 \) from equation (5.3.7), (5.3.8) and (5.3.11), (5.3.12) we get

**Region-I**

\[
u_0 \left( \frac{\partial^3 \psi_1}{\partial x \partial y^2} + \frac{\partial^3 \psi_1}{\partial x^3} \right) - \frac{\partial \psi_1}{\partial x} \frac{d^2 u_0}{dy^2} = 2 \frac{\partial^3 \psi_1}{\partial x^2 \partial y^2} + \frac{\partial^3 \psi_1}{\partial x^4} - \frac{\partial^3 \psi_1}{\partial x^2 \partial y^4} - G \frac{\partial \theta_0}{\partial y} - \frac{\partial^2 \theta_1}{\partial y^2}
\]

**Region-II**

\[
u_0 \left( \frac{\partial^3 \psi_2}{\partial x \partial y^2} + \frac{\partial^3 \psi_2}{\partial x^3} \right) - \frac{\partial \psi_2}{\partial x} \frac{d^2 u_0}{dy^2} = 2 \frac{\partial^3 \psi_2}{\partial x^2 \partial y^2} + \frac{\partial^3 \psi_2}{\partial x^4} + \frac{\partial^3 \psi_2}{\partial x^2 \partial y^4} - \beta m^2 r^2 h G \frac{\partial \theta_2}{\partial y} - m h^2 M^2 \frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial^2 \theta_1}{\partial y^2}
\]

We assume the stream function in the form

\[
\psi (x, y) = e^{i \lambda x} \psi (y), \quad \theta_1 (x, y) = e^{i \lambda t} \theta_1 (y)
\]

From which we infer

\[
u_1 (x, y) = e^{i \lambda x} \nu_1 (y), \quad v_1 (x, y) = e^{i \lambda t} v_1 (y)
\]
Below we restrict our attention to the real parts of the solution for the perturbed quantities $\varphi^{(1)}_i, \theta^{(1)}_i, u^{(1)}_i$ and $v^{(1)}_i$ for $i = 1, 2$

The boundary conditions (5.3.17) and (5.3.18) can be now written in terms of $\varphi^{(1)}_i$ as

\[
\frac{\partial \varphi^{(1)}_i}{\partial y} = -\cos \lambda x \frac{d u^{(1)}_i}{d y}; \quad \frac{\partial \varphi^{(1)}_i}{\partial x} = 0; \quad \text{at} \quad y = -1
\]

\[
\frac{\partial \varphi^{(2)}_i}{\partial y} = 0; \quad \frac{\partial \varphi^{(2)}_i}{\partial x} = 0; \quad \text{at} \quad y = 1
\]

\[
\frac{\partial \varphi^{(1)}_i}{\partial y} = \frac{1}{r m h} \frac{\partial \varphi^{(2)}_i}{\partial x}; \quad \frac{\partial \varphi^{(2)}_i}{\partial y} = \frac{1}{r m h} \frac{\partial \varphi^{(2)}_i}{\partial x}; \quad \text{at} \quad y = 0
\]

\[
\frac{\partial^3 \varphi^{(1)}_i}{\partial x^3} - \frac{\partial^3 \varphi^{(1)}_i}{\partial y^3} = \frac{1}{r m^2 h^2} \left( \frac{\partial^3 \varphi^{(2)}_i}{\partial x^2} - \frac{\partial^3 \varphi^{(2)}_i}{\partial y^2} \right)
\]

\[
= \frac{1}{r m^2 h^2} \left( -u^{(2)}_i \frac{\partial^2 \varphi^{(2)}_i}{\partial x^2} + \frac{\partial \varphi^{(2)}_i}{\partial x} u^{(2)}_i + \frac{\partial^3 \varphi^{(2)}_i}{\partial x^3} - Gr \beta^i \right)
\]

\[
= \frac{1}{r m^2 h^2} \left( -u^{(2)}_i \frac{\partial^2 \varphi^{(2)}_i}{\partial x^2} + \frac{\partial \varphi^{(2)}_i}{\partial x} u^{(2)}_i + \frac{\partial^3 \varphi^{(2)}_i}{\partial x^3} - Gr \beta^i \right) - G r \beta^i m^3 r^2 \theta^{(2)}_i
\]

\[
i^{(1)} = -\cos (\lambda x) \frac{d \theta^{(1)}_i}{d y}
\]

\[
i^{(2)} = 0
\]

\[
i^{(1)} = i^{(2)}
\]

\[
\frac{d^2 i^{(1)}}{d x^2} + \frac{d^2 i^{(1)}}{d y^2} = \frac{k}{h} \left( \frac{d^2 i^{(2)}}{d x^2} + \frac{d^2 i^{(2)}}{d y^2} \right)
\]

\[
\text{at} \quad y = 0
\]
If we consider small values of $\lambda$ then substituting

$$\psi^{(i)}(\lambda, y) = \psi_0^{(i)} + \lambda \psi_1^{(i)} + \lambda^2 \psi_2^{(i)} + \ldots$$

$$i^{(i)}(\lambda, y) = i_0^{(i)} + \lambda i_1^{(i)} + \lambda^2 i_2^{(i)} + \ldots \quad \text{for } i = 1, 2 \quad (52)$$

In to (5.3.26) to (5.3.31) gives, to order of $\lambda$, the following sets of ordinary differential equations and corresponding boundary and conditions

**Region-I**

$$\frac{d^4 \psi_0^{(i)}}{dy^4} = G \frac{d i_0^{(i)}}{dy} \quad (53)$$

$$\frac{d^4 i_0^{(i)}}{dy^4} = 0 \quad (54)$$

$$\frac{d^4 \psi_1^{(i)}}{dy^4} - i \left( u_0 \frac{d^2 \psi_0}{dy^2} - \frac{d^2 u_0}{dy^2} \psi_0 \right)^{(i)} = G \frac{d i_1^{(i)}}{dy} \quad (55)$$

$$\frac{d^4 i_1^{(i)}}{dy^4} = \text{Pr} \ i \left( u_0 i_0 + \psi_0 \frac{d \psi_0}{dy} \right)^{(i)} \quad (56)$$

**Region-II**

$$\frac{d^4 \psi_0^{(2)}}{dy^4} = \beta m^2 r^2 h^3 G \frac{d i_0^{(2)}}{dy} \quad (57)$$

$$\frac{d^4 i_0^{(2)}}{dy^4} = 0 \quad (58)$$

$$\frac{d^4 \psi_1^{(2)}}{dy^4} - M^2 m h^2 \frac{d^2 \psi_1^{(2)}}{dy^2} = i \left( u_0 \frac{d^2 \psi_0}{dy^2} - \frac{d^2 u_0}{dy^2} \psi_0 \right)^{(2)} + \beta m^2 r^2 h^3 G \frac{d i_1^{(2)}}{dy} \quad (59)$$

$$\frac{d^4 i_1^{(2)}}{dy^4} = \frac{\text{Pr}}{km} i \left( u_0 i_0 + \psi_0 \frac{d \psi_0}{dy} \right)^{(2)} \quad (60)$$

with boundary conditions

$$\frac{d \psi_0^{(1)}}{dy} = \cos \lambda x \frac{d i_0^{(1)}}{dy}; \quad \psi_0^{(1)} = 0 \quad \text{at } y = -1$$

$$\frac{d \psi_0^{(2)}}{dy} = 0; \quad \psi_0^{(2)} = 0 \quad \text{at } y = 1$$

$$\frac{d \psi_0^{(1)}}{dy} = \frac{1}{rmh} \frac{d \psi_0^{(2)}}{dy}; \quad \psi_0^{(1)} = \frac{1}{rmh} \psi_0^{(2)} \quad \text{at } y = 0 \quad \frac{d^2 \psi_0^{(1)}}{dy^2} = \frac{1}{rm^2 h^2} \frac{d^2 \psi_0^{(2)}}{dy^2}; \quad (61)$$

$$\psi_0^{(1)} = \frac{1}{rm^2 h^2} \psi_0^{(2)} \quad \text{at } y = 0$$
\[
\begin{align*}
\frac{d\psi_1^{(1)}}{dy} = 0; & \quad \psi_1^{(1)} = 0 \quad \text{at} \quad y = -1 \\
\frac{d\psi_1^{(2)}}{dy} = 0; & \quad \psi_1^{(2)} = 0 \quad \text{at} \quad y = 1 \\
\frac{d\psi_1^{(1)}}{dy} = \frac{1}{r m h} \frac{d\psi_1^{(2)}}{dy}; & \quad \psi_1^{(1)} = \frac{1}{r m h} \psi_1^{(2)} \quad \text{at} \quad y = 0 \\
d^2\psi_1^{(1)} + \lambda^2\psi_1^{(1)} = \frac{1}{r m h^2} \frac{d^2\psi_1^{(2)}}{dy^2} + \lambda^2\psi_1^{(2)}; \\
-\imath\lambda u_0^{(1)} \psi_y^{(1)} + \imath\lambda \psi^{(1)} u_0^{(1)} - \lambda^2 \psi^{(1)} + \psi_y^{(1)} - G \epsilon^{(1)} \\
= \frac{1}{r m h^2} \left( -\imath\lambda u_0^{(2)} \psi_y^{(2)} + \imath\lambda \psi^{(2)} u_0^{(2)} - \lambda^2 \psi_y^{(2)} + \psi_y^{(2)} - G \beta m^2 r^2 \psi_y^{(2)} \right) \quad \text{at} \quad y = 0
\end{align*}
\]

(62)

\[
\begin{align*}
i_0^{(1)} = -\frac{d\theta_0^{(1)}}{dy} \quad \text{at} \quad y = -1 \\
i_0^{(2)} = 0 \quad \text{at} \quad y = 1 \\
i_0^{(1)} = i_0^{(2)} \quad \text{at} \quad y = 0 \\
\frac{dt_0^{(1)}}{dy} + i\lambda t^{(1)} = k \left( \frac{dt_0^{(2)}}{dy} + i\lambda t^{(2)} \right) \quad \text{at} \quad y = 0
\end{align*}
\]

(63)

\[
\begin{align*}
i_1^{(1)} = 0 \quad \text{at} \quad y = -1 \\
i_1^{(2)} = 0 \quad \text{at} \quad y = 1 \\
i_1^{(1)} = i_1^{(2)} \quad \text{at} \quad y = 0 \\
\frac{dt_1^{(1)}}{dy} = \frac{dt_1^{(2)}}{dy} \quad \text{at} \quad y = 0
\end{align*}
\]

(64)

**Zeroth-order solution (mean part)**

The solutions to zeroth order differential Eqs. (5.3.3) to (5.3.6) using boundary and interface conditions (5.3.15) and (5.3.16) are given by

**Region-I**

\[
\begin{align*}
u_0^{(1)} &= l_1 y^3 + l_2 y^2 + A_1 y + A_2 \\
\theta_0^{(1)} &= C_1 y + C_2
\end{align*}
\]

**Region-II**

\[
\begin{align*}
u_0^{(2)} &= B_1 \cosh ny + B_2 \sinh ny + s_1 y^3 + s_1 y + s_2 \\
\theta_0^{(2)} &= C_3 y + C_4
\end{align*}
\]
The solutions of zeroth and first order of $\lambda$ are obtained by solving the equation (5.3.33) to (5.3.40) using boundary and interface conditions (5.3.41) and (5.3.42) and are given below

$$\psi_0^{(0)} = I_4 y^4 + \frac{A_1}{6} y^3 + \frac{A_2}{2} y^2 + A_3 y + A_4$$

$$t_0^{(1)} = C_5 y + C_6$$

$$\psi_0^{(2)} = B_3 \cosh ny + B_4 \sinh ny + B_5 y + B_6 + s_4 y^2$$

$$t_0^{(2)} = C_7 y + C_8$$

$$\psi_1^{(1)} = i \left( l_{00} y^{10} + l_{01} y^9 + l_{02} y^8 + l_{03} y^7 + l_{04} y^6 + l_{05} y^5 + l_{06} y^4 \right) + \frac{A_1}{6} y^3 + \frac{A_2}{2} y^2 + A_3 y + A_4 + A_5 y + A_6$$

$$t_1^{(1)} = i \Pr \left( d_8 y^7 + d_9 y^6 + d_{10} y^5 + d_{11} y^4 + d_{12} y^3 + d_{13} y^2 \right) + q_6 y + q_7$$

$$\psi_1^{(2)} = B_1 \cosh ny + B_4 \sinh ny + B_5 y + B_6 + i \left( s_{29} y^3 \cosh ny + s_{30} y^3 \sinh ny + s_{31} y^2 \cosh ny + s_{32} y^2 \sinh ny + s_{33} y \cosh ny + s_{34} y \sinh ny + s_{35} \cosh hny + s_{36} \sinh hny + S_{37} y^6 + S_{38} y^5 + S_{39} y^4 + S_{40} y^3 + S_{41} y^2 \right)$$

$$t_1^{(2)} = \frac{i \Pr}{k m} \left( f_{11} y \cosh ny + f_{12} y \sinh ny + f_{13} \cosh ny + f_{14} \sinh ny + f_{15} y^5 + f_{16} y^4 \right) + f_{17} y^3 + f_{18} y^2 \right) + q_5 y + q_7$$

The first order quantities can be put in the forms

$$u_1 = \psi_i' \sin \lambda x - \psi_r' \cos \lambda x$$

$$v_1 = \lambda \psi_i' \sin \lambda x - \lambda \psi_r' \cos \lambda x$$

$$\theta_1 = t_r \cos \lambda x - t_i \sin \lambda x$$

**Region-I**

$$u_1^{(1)} = -\cos \lambda x (\psi_{0r} + \lambda \psi_{1r})^{(1)} + \sin \lambda x (\psi_{0i} + \lambda \psi_{1i})^{(1)}$$

$$v_1^{(1)} = -\lambda \cos \lambda x (\psi_{0r} + \lambda \psi_{1r})^{(1)} - \lambda \sin \lambda x (\psi_{0i} + \lambda \psi_{1i})^{(1)}$$

$$\theta_1^{(1)} = \cos \lambda x (t_{0r} + \lambda t_{1r})^{(1)} - \sin \lambda x (t_{0i} + \lambda t_{1i})^{(1)}$$

**Region-II**

$$u_1^{(2)} = -\cos \lambda x (\psi_{0r} + \lambda \psi_{1r})^{(2)} + \sin \lambda x (\psi_{0i} + \lambda \psi_{1i})^{(2)}$$

$$v_1^{(2)} = -\lambda \cos \lambda x (\psi_{0r} + \lambda \psi_{1r})^{(2)} - \lambda \sin \lambda x (\psi_{0i} + \lambda \psi_{1i})^{(2)}$$

$$\theta_1^{(2)} = \cos \lambda x (t_{0r} + \lambda t_{1r})^{(2)} - \sin \lambda x (t_{0i} + \lambda t_{1i})^{(2)}$$
Skin friction and Nusselt number

The shearing stress $\tau_{xy}$ at any point in the fluid is given in nondimensional form, by

$$\tau_{xy} = \left( \frac{h^2}{\rho \nu^2} \right) \tau_{xy} = u_0'(y) + \varepsilon e^{i\lambda x} u_1'(y) + i \varepsilon \lambda e^{i\lambda x} v_1(y)$$

$\tau_{xy}$ at the wavy wall ($y = -h^{(1)} + \varepsilon \cos \lambda x$) and at the flat wall $y = 1$ are given by

$$\tau^{(1)} = \tau_0^{(1)} + \varepsilon \Re \left[ e^{i\lambda x} (u_0^* + u_1^*) \right]^{(1)}$$
$$\tau^{(2)} = \tau_0^{(2)} + \varepsilon \Re \left[ e^{i\lambda x} u_1^* \right]^{(2)}$$

Where $\Re$ represent the real part of

$$\tau_0^{(1)} = -4 l_1 + 3 l_2 - 2 l_3 + l_4$$
$$\tau_0^{(2)} = B_2 n \sin h n + B_3 n \cosh n + 2 s_1 + s_2$$
$$\tau_1^{(1)} = u_0^{(1)} + \cos \lambda x u_0^{(1)} + \left( -\psi_{0r}^* \cos \lambda x + \lambda \psi_{li}^* \sin \lambda x \right)^{(1)}$$
$$\tau_1^{(2)} = u_0^{(2)} + \left( -\psi_{0r}^* \cos \lambda x + \lambda \psi_{li}^* \sin \lambda x \right)^{(2)}$$

The nondimensional Nusselt number is given by

$$Nu = \frac{d\theta}{dy} = \theta_0'(y) + \varepsilon e^{i\lambda x} \theta'(y)$$

At the wavy wall and flat flat wall Nu takes the form

$$Nu^{(1)} = Nu_0^{(1)} + \varepsilon \Re \left[ e^{i\lambda x} (\theta_0^* + t_1^*) \right]^{(1)}$$
$$Nu^{(2)} = Nu_0^{(2)} + \varepsilon \Re \left[ e^{i\lambda x} t_1^* \right]^{(2)}$$

where

$$Nu_0^{(1)} = -2 d_1 + C_1$$
$$Nu_0^{(2)} = 2 f_2 + C_3$$

Pressure drop:

The fluid pressure $\bar{P}(x, y)$ ( $P_0(x) = \text{constant}$) at any point $(x, y)$ as

$$\bar{P}(x, y) = \int d\bar{P} = \int \left[ \frac{\partial \bar{P}}{\partial x} dx + \frac{\partial \bar{P}}{\partial y} dy \right]$$
\[ i.e \quad \overline{P}(x, y) - L = \text{Re} \left[ e^{\frac{i\lambda x}{\lambda}} \right] \]

where \( L \) is an arbitrary constant and

\[
Z(y) = (\psi'' - \lambda^2 \psi') - i\lambda (u_0 \psi' - u_0' \psi) - G t
\]

\[
Z(y) = (\psi_0'' + \lambda \psi_0') - \lambda^2 (\psi_0' + \lambda \psi_1') - i\lambda (u_0 (\psi_0' + \lambda \psi_1') - u_0' (\psi_0 + \lambda \psi_1)) - G (t_0 + \lambda t_1)
\]

Equation (5.3.45) can be rewritten as

\[
\hat{P} = \overline{P}(x, y) - \overline{P}(x, 1) = (\varepsilon / \lambda) \text{Re}\left[ i e^{i\lambda x} (z(y) - z(1)) \right]
\]

\[
\overline{P} = \frac{\varepsilon}{\lambda} \text{Re}\left[ i (\cos \lambda x + i \sin \lambda x)(z(y) - z(1)) \right]
\]

\[
= \frac{\varepsilon}{\lambda} \text{Re}\left[ i (\cos \lambda x - \sin \lambda x)(z(y) - z(1)) \right]
\]

where \( \hat{P} \) has been named the pressure drop since it indicates the difference between the present at any point \( y \) in the flow field and that at the flat wall, with \( x \) fixed. The pressure drops \( \hat{P} \) at \( \lambda \chi = 0 \) and \( 1/2 \pi \) have been named \( \hat{P}_0 \) and \( \hat{P}_{\pi/2} \) and their numerical values for several sets of values of the non-dimensional parameters \( G \) have been evaluated. In what follows we record the qualitative differences in the behavior of the various flow and heat-transfer characteristics which show clearly the effects of the wavy wall of the channel under consideration.

4. RESULTS AND DISCUSSION

Discussion of the Zeroth Order Solution

The effect of Hartmann number \( M \) on zeroth order velocity is to decrease the velocity for \( \varepsilon = 0, \pm 1 \), but the suppression is more effective near the flat wall as \( M \) increases as shown in figure 2. The effect of heat source \( (\alpha > 0) \) or sink \( (\alpha < 0) \) and in the absence of heat source or sink \( (\alpha = 0) \), on zeroth order velocity is shown in figure 3 for \( \varepsilon = 0, \pm 1 \). It is observed that heat source promote the flow, sink suppress the flow, and the velocity profiles lie in between source or sink for \( \alpha = 0 \). We also observe that the magnitude of zeroth order velocity is optimum for \( \varepsilon = 1 \) and minimal for \( \varepsilon = -1 \), and profiles lies between \( \varepsilon = \pm 1 \) for \( \varepsilon = 0 \). The effect of source or sink parameter \( \alpha \) on zeroth order temperature is similar to that on zeroth order velocity as shown in figure 4. The effect of free convection parameter \( G \), viscosity ratio \( m \), width ratio \( h \), on zeroth order velocity and the effect of width ratio \( h \), conductivity ratio \( k \), on zeroth order temperature remain the same as explained in chapter-III

The effect of free convection parameter \( G \) on first order velocity is shown in figure 5. As \( G \) increases \( u_1 \) increases near the wavy and flat wall where as it decreases at the interface and the suppression is effective near the flat wall for \( \varepsilon = 0, \pm 1 \). The effect of viscosity ratio \( m \) on \( u_1 \) shows that as \( m \) increases first order velocity increases near the wavy and flat wall, but the magnitude is very large near flat wall. At the interface velocity decreases as \( m \) increases and the suppression is significant towards the flat wall as seen in figure 6 for \( \varepsilon = 0, \pm 1 \). The effect of width ratio \( h \) on first
order velocity $u_1$ shows that $u_1$ remains almost same for $h<1$ but is more effective for $h>1$. For $h = 2$, $u_1$ increases near the wavy and flat wall and drops at the interface, for $\theta = 0, \pm 1$ as seen in figure 7. The effect of Hartmann number $M$ on $u_1$ shows that as $M$ increases velocity decreases at the wavy wall and the flat wall but the suppression near the flat wall compared to wavy wall is insignificant and as $M$ increases $u_1$ increases in magnitude at the interface for $\theta = 0, \pm 1$ as seen in figure 8. The effect of $\alpha$ on $u_1$ is shown in figure 9, which shows that velocity is large near the wavy and flat wall for heat source $\alpha = 5$ and is less for heat sink $\alpha = -5$. Similar result is obtained at the interface but for negative values of $u_1$. Here also we observe that the magnitude is very large near the wavy wall compared to flat wall.

The effect of convection parameter $G$, viscosity ratio $m$, width ratio $h$, thermal conductivity ratio $k$ on first order velocity $v_1$ is similar. The effect of Hartmann number $M$ is to increase the velocity near the wavy wall and decrease velocity near the flat wall for $\theta = 0, \pm 1$, whose results are applicable to flow reversal problems as shown in figure 10. The effect of source or sink on first order velocity $v_1$ is shown in figure 11. For heat source, $v_1$ is less near wavy wall and more for flat wall where as we obtained the opposite result for sink i.e. $v_1$ is maximum near wavy wall and minimum near the flat wall, for $\alpha = 0$ the profiles lie in between $\alpha = \pm 5$.

The effect of convection parameter $G$, viscosity ratio $m$, width ratio $h$, thermal conductivity ratio $k$, Hartmann number and source and sink parameter $\alpha$ on first order temperature are shown in figures 12 to 17. It is seen that $G$, $m$, $h$, and $k$ increases in magnitude for values of $\theta = 0, \pm 1$. It is seen that from figure 16 that as $M$ increases the magnitude of $\theta_1$ decreases for $\theta = 0, \pm 1$. Figure17 shows that the magnitude of $\alpha$ is large for heat source and is less for sink where as $\theta_1$ remains invariant for $\alpha = 0$.

The effect of convection parameter $G$, viscosity ratio $m$, width ratio $h$, and thermal conductivity ratio $k$, on total velocity remains the same. The effect of heat source or sink on total velocity shows that $U$ is very large for $\alpha = 5$ compared to $\alpha = -5$ and is almost invariant for $\alpha = 0$ as seen figure 18, for all values of $\theta$.

The effect of Grashof number $G$, viscosity ratio $m$, shows that increasing $G$ and $m$ suppress the total temperature but the supression for $m$ is negligible as seen in figures 19 and 20. The effects of width ratio $h$ and conductivity ratio $k$ is same. The effect of source or sink parameter on total temperature remains the same as that on total velocity as seen figure 21.
**Figure 16** First order solution of velocity profile for different values of Hartmann number $M$

**Figure 17** First order solution of velocity profile for different values of $\alpha$

**Figure 21** First order solution of velocity profile for different values of $k$

**Figure 22** First order solution of velocity profile for different values of Hartmann number $M$

**Figure 23** First order solution of velocity profile for different values of $\alpha$

**Figure 27** First order solution of temperature profile for different values of $k$

**Figure 28** First order solution of temperature profile for different values of Hartmann number $M$

**Figure 29** First order solution of temperature profile for different values of $\alpha$

**Figure 30** Total solution of velocity profile for different values of Grashof number $Gr$
Fig 31: Total solution of velocity profiles for different values of viscosity ratio m

Fig 32: Total solution of velocity profiles for different values of width ratio h

Fig 33: Total solution of velocity profiles for different values of k

Fig 34: Total solution of velocity profiles for different values of Hartmann number M

Fig 35: Total solution of velocity profiles for different values of x

Fig 36: Total solution of temperature profiles for different values of Grashof number Gr

Fig 37: Total solution of temperature profiles for different values of viscosity ratio m

Fig 38: Total solution of temperature profiles for different values of width ratio h

Fig 39: Total solution of temperature profiles for different values of k
Fig. 40 Total solution of temperature profile for different values of Hartmann number $M$.

Fig. 41 Total solution of temperature profile for different values of $\alpha$.

Fig. First order Nusselt number at $y = -1.0$ and $y = 1.0$.

Fig. Zeroth order Nusselt number at $y = -1.0$ and $y = 1.0$.

Fig. Zeroth order Nusselt number at $y = -1.0$ and $y = 1.0$.

Fig. Zeroth order Nusselt number at $y = -1.0$ and $y = 1.0$.

Fig. First order skin friction profiles for different values of $h$.

Fig. First order skin friction profiles for different values of $K$.

Fig. First order skin friction profiles for different values of $Gr$.
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